Math Olympics 2016 : Round 1

Math and Physics Club, IIT Bombay

Time: 45 minutes

Name: E-mail:

---

1. If the sum of the roots of the equation $2^{333x-2} + 2^{111x+1} = 2^{222x+2} + 1$ is expressed in it's lowest form as S_1/S_2, find $S_1 + S_2$. **[10]**
   Solution:
   Replace 2^{111x} with t to get a cubic equation in t.
   $$\frac{t^3}{4} - 4t^2 + 2t + 1 = 0$$
   We see that we must find the sum of roots of possible x, which would translate into the product of roots of the possible t's.
   The sum of roots of x is thus $\frac{2}{111}$, giving $S_1 + S_2 = 113$.

2. For $a \geq 2$, if the value of the definite integral
   $$\int_0^\infty \frac{dx}{a^2 + (x - \frac{1}{x})^2}$$
   equal $\frac{\pi}{5050}$, find the value of a. **[10]**
   Solution:
   Let the required integral be I. Putting $t = \frac{1}{x}$, we get:
   $$I = \int_0^\infty \frac{\frac{dt}{t^2}}{a^2 + (t - \frac{1}{t})^2}$$
   Adding and putting $x - \frac{1}{x} = \alpha$, we get
   $$2I = \int_{-\infty}^\infty \frac{d\alpha}{a^2 + \alpha^2} = \frac{\pi}{a}$$
   Thus we get $a = 2525$

3. T_1 is an isosceles $\triangle$ with circumcircle k. Let T_2 be another isosceles $\triangle$ inscribed in k whose base is one of the equal sides of T_1 and which overlaps the interior of T_1. Similarly create isosceles $\triangle\mathrm{T}_3$ and so on. Comment on the nature of $\triangle\mathrm{T}_n$, $n \to \infty$. **[10]**
   Solution:
   Observe either the angles or the side ratio. The triange converges to an equilateral triangle.

4. Let n be a natural number. Prove that

$$\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+...+\left[\frac{n}{n}\right]+\left[\sqrt{n}\right]$$

is even. (Here $[x]$ denotes the largest integer smaller than or equal to x). **[10]**
Solution:
We make the following observations:

$\left[\frac{n+1}{r}\right]=\left[\frac{n}{r}\right] \iff r$ does not divide $(n+1)$ while,
$\left[\frac{n+1}{r}\right]=\left[\frac{n}{r}\right]+1 \iff r$ divides $(n+1)$

and

$\left[\sqrt{n+1}\right]=\left[\sqrt{n}\right] \iff (n+1)$ is not a perfect square while,
$\left[\sqrt{n+1}\right]=\left[\sqrt{n}\right]+1 \iff (n+1)$ is a perfect square

Further,
Number of positive divisors of $n+1$ *is odd* $\iff$ $(n+1)$ *is a perfect square*
Putting these three observations together, and handling the trivial base case for $n=1$, the induction argument can easily be completed.

5. Prove that the equation $4a^3+2b^3=c^3$ does not have any solutions over $\mathbb{N}$. **[15]**
Solution:
Let us assume that a solution exists. Then, the smallest a, b, c are being considered (no common factors. Such a set always exists.) here onwards. Since they have no common factors, they can not be *all even.*
Observe that RHS must be even because it is sum of two even quantitites. Thus $c=2k_c$ for some k_c.
Arranging quantities and pairing up terms with a and c, we get $2b^3$ is a difference of two factors of 4 and hence b must be even. By a similar argument we get a to be even and the assumption is violated. *Hence, no such set* $\{a, b, c\}$ *can be formed over* $\mathbb{N}$

6. A point X is chosen inside or on a circle. Two perpendicular chords AC & BD of the circle are drawn through X. (In the case when X is on the circle, the degenerate case, when one chord is a diameter and the other is a point, is allowed.) Find the greatest & least values which the sum $S=|AC|+|BD|$ can take for all choices of X. **[10]**
Solution:

We can choose an (x,y) coordinate system such that the centre if the circle coincides with the origin. Without loss of generality let BD be parallel to the x-axis and AC be parallel to the y- axis. Lets (r, s) be the intersection of the two lines. For any (r, s) lying in the circle we can increase the sum by translating any one line towards the origin. Therefore the maiximum will be when $(r, s) = (0, 0)$ with a value of $4R$
For the minimum,
we have $S(r, s) = |AC| + |BD| = 2(\sqrt{R^2 - r(2)} + \sqrt{R^2 - s^2})$
where R is the radius of the circle. By minimizing the function we get

$$(\sqrt{R^2 - r(2)} + \sqrt{R^2 - s^2})^2 \geq 2R^2 - (r^2 + s^2) \geq R^2$$

Therefore at minimum $S = 2R$

7. Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation

$$f(x + a) = \frac{1}{2} + \sqrt{f(x) - (f(x))^2}$$

holds for all x. Prove that f is periodic,ie, $f(x + b) = f(x) \; \forall \; x$. **[10]**
Solution:
Since $f(x + a) \geq \frac{1}{2}$ for all x, and

$$f(x + a)(1 - f(x + a)) = \frac{1}{4} - (f(x) - f(x)^2) = (\frac{1}{2} - f(x))^2$$

Thus we have,

$$f(x + 2a) = \frac{1}{2} + \sqrt{(\frac{1}{2} - f(x))^2} = f(x)$$

since $f(x) \geq \frac{1}{2}$. This gives f to be periodic with $2a > 0$ as the period.

8. Prove that for every $n \in \mathbb{Z}^+$, there exists a multiple of n whose decimal expansion only has 0's and 1's. **[10]**
Solution
For $r = 1, 2, \ldots, n + 1$, let $a_r = (10^r - 1)/9 = 11\ldots1$ (1 occurs r times). There are at most n possible remainders when a positive integer is divided by n. Hence by PHP, $\exists \; r_1, r_2$ such that a_{r_1} and a_{r_2} leave the same remainder when divided by n and $r_1 < r_2$. Clearly, $a_{r_2} - a_{r_1}$ is a multiple of n and consists only of 1's and 0's in it's decimal expansion.

9. Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2. **[15]**
Solution:

Let $P = x_1x_2x_3x_4$ be the product of the four real numbers.

Then, for $i = 1, 2, 3, 4$ we have: $x_i + \prod_{j \neq i} x_j = 2$.

Multiplying by x_i yields:

$$x_i^2 + P = 2x_i \iff x_i^2 - 2x_i + 1 = (x_i - 1)^2 = 1 - P \iff x_i = 1 \pm t$$

where $t = \pm\sqrt{1 - P} \in \mathbb{R}$.

If $t = 0$, then we have $(x_1, x_2, x_3, x_4) = (1, 1, 1, 1)$ which is a solution.

So assume that $t \neq 0$. WLOG, let at least two of x_i equal $1 + t$, and $x_1 \geq x_2 \geq x_3 \geq x_4$ OR $x_1 \leq x_2 \leq x_3 \leq x_4$.

Case I: $x_1 = x_2 = x_3 = x_4 = 1 + t$

Then we have:

$$(1 + t) + (1 + t)^3 = 2 \iff t^3 + 3t^2 + 4t = 0 \iff t(t^2 + 3t + 4) = 0$$

Which has no non - zero solutions for t.

Case II: $x_1 = x_2 = x_3 = 1 + t$ AND $x_4 = 1 - t$

Then we have:

$$(1 - t) + (1 + t)^3 = 2 \iff t^3 + 3t^2 + 2t = 0$$
$$\iff t(t + 1)(t + 2) = 0 \iff t \in \{0, -1, -2\}$$

AND

$$(1 + t) + (1 - t)(1 + t)^2 = 2(1 + t) + (1 - t)(1 + t)^2 = 2 - t^3 - t^2 + 2t = 0$$
$$\iff -t(t - 1)(t + 2) = 0 \iff t \in \{0, 1, -2\}$$

So, we have $t = -2$ as the only non-zero solution, and thus, $(x_1, x_2, x_3, x_4) = (-1, -1, -1, 3)$ and all permutations are solutions.

Case III: $x_1 = x_2 = 1 + t$ AND $x_3 = x_4 = 1 - t$

Then we have:

$$(1-t) + (1-t)(1+t)^2 = 2 \Longleftrightarrow -t^3 - t^2 = 0$$
$$\Longleftrightarrow -t^2(t+1) = 0 \Longleftrightarrow t \in \{0, -1\}$$

And $$(1+t) + (1+t)(1-t)^2 = 2 \Longleftrightarrow t^3 - t^2 = 0$$
$$\Longleftrightarrow t^2(t-1) = 0 \Longleftrightarrow t \in \{0, 1\}$$

Thus, there are no non-zero solutions for t in this case.Therefore, the solutions are:

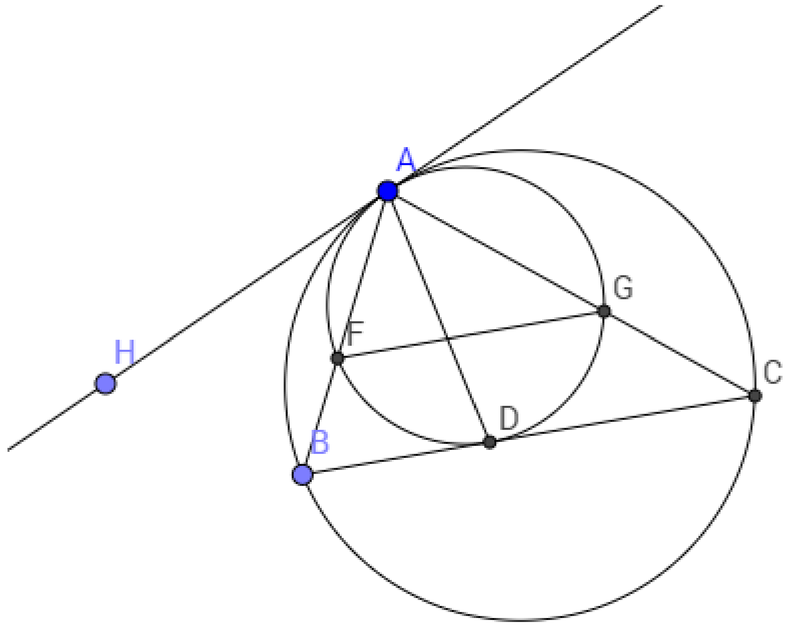
$(1,1,1,1)$; $(3,-1,-1,-1)$; $(-1,3,-1,-1)$; $(-1,-1,3,-1)$; .

10. A bag contains one red and one blue disc. In a game a player pays £2 to play and takes a disc at random. If the disc is blue then it is returned to the bag, one extra red disc is added, and another disc is taken. This game continues until a red disc is taken. Once the game stops the player receives winnings equal in pounds to the number of red discs that were in the bag on that particular turn. Find the exact value of the expected return on this game. **[10]**

Solution

Let X be the number of red discs in the bag when the play stops,
$P(X=1) = \frac{1}{2}$
$P(X=2) = \frac{1}{2} \times \frac{2}{3}$
$P(X=3) = \frac{1}{2} \times \frac{1}{3} \times \frac{3}{4}$
...
$P(X=k) = \frac{1}{2} \times \frac{1}{3} \times ... \times \frac{k}{k+1} = \frac{k}{(k+1)!}$
$\rightarrow E(X) = \frac{1^2}{2!} + \frac{2^2}{3!} + \frac{3^2}{4!} + ...$To evaluate this, consider the term $\frac{k^2}{(k+1)!}$. Writing it as $\frac{1}{(k-1)!} - \frac{k}{(k+1)!}$ we get
$E(X) = e - 1$
Hence the return on each game would be $2 - (e-1) \approx 28$ pence in the banker's favour.

11. Two circles touch internally at A. Chord BC of outer circle is tangent to inner circle at D. Prove that AD bisects $\angle BAC$. **[10]**



Solution:

$\angle HAB = \angle AGF$ and $\angle HAB = \angle ACB \Rightarrow FG \parallel BC \Rightarrow \frac{BF}{BA} = \frac{CG}{CA} \Rightarrow \frac{BF}{CG} = \frac{BA}{CA} \ldots 1$

BD is tangent to the smaller circle at D $\Rightarrow BD^2 = BF \cdot BA$

Similarly, $CD^2 = CG \cdot CA$

Hence, $\frac{BD^2}{CD^2} = \frac{BF \cdot BA}{CG \cdot CA} = \frac{BA^2}{CA^2}$ (by 1) $\Rightarrow \frac{BD}{CD} = \frac{BA}{CA} \Rightarrow AD$ bisects $\angle BAC$