Math Olympics 2016 : Round 2

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Time: 45 minutes

Name: E-mail:

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1. Given that m and n are distinct positive integers, solve $m^n = n^m$. **[25]**

Solution

Without loss of generality assume $m < n$. Notice that $m = 1 \implies n = 1$ and hence $m \neq 1$. Thus $2 \leq m < n$.
$m^n = n^m \implies m^{n-m} = (\frac{n}{m})^m$
Since LHS is an integer, the RHS is also an integer. Hence $\frac{n}{m} = k \in \mathbf{N}$ and $k \geq 2$
Simplify the original equation to get $m^{k-1} = k$. Observe that $k = 2 \implies m = 2$ and this is a solution. By induction it is easy to prove that $m^{k-1} > k \quad \forall\, k \geq 3$. Hence the only solution to the original equation is $(m, n) = (2, 4)$.

2.

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x + 3}{3\sin x + 4\cos x + 25}\, dx$$

Find I. **[20+10*]**

Solution

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x + 3}{3(\sin x + 3) + 4(\cos x + 4)} dx \qquad \text{and} \qquad J = \int_0^{\frac{\pi}{2}} \frac{\cos x + 4}{3(\sin x + 3) + 4(\cos x + 4)} dx$$

$$\implies 3I + 4J = \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2}$$

and

$$3J - 4I = \int_0^{\frac{\pi}{2}} \frac{3\cos x - 4\sin x}{3(\sin x + 3) + 4(\cos x + 4)} dx = \ln\frac{28}{29}$$

$$\implies I = \frac{1}{25}(\frac{3\pi}{2} + 4\ln\frac{28}{29})$$

3. In a sports contest, there were m medals awarded on n successive days ($n > 1$). On the first day, one medal and $\frac{1}{7}$ of the remaining $m-1$ medals were awarded. On the second day, two medals and $\frac{1}{7}$ of the now remaining medals were awarded; and so on. On the n^{th} and last day, the remaining n medals were awarded. How many days can the contest last, and how many medals were awarded altogether? **[25]**

Solution

Let a_i be the number of medals undistributed at the start of the i^{th} day. We have

$$a_1 = m,\ a_{m+1} = 0$$

$$a_i - a_{i-1} = i + \frac{1}{7}(a_i - i)$$

$$\implies 6a_i - 7a_{i-1} = 6i$$

$$6(a_i + 6i - 42) = 7(a_{i+1} + 6(i+1) - 42)$$

We notice that for $x_i = a_i + 6i - 42$, $6x_i = 7x_{i+1}$

$$\implies a_i + 6i - 42 = \left(\frac{6}{7}\right)^{i-1}(a_1 + 6 - 42)$$

We know $a_{n+1} = 0$ and $a_1 = m$

$$\implies 6(n+1) - 42 = \left(\frac{6}{7}\right)^{n}(m - 36)$$

$$\implies 7^n(n-6) = 6^{n-1}(m-36)$$

$$6^{n-1}|n-6$$

But, $6^{n-1} > (n-6)$ only for $n > 6$ and hence

$$\implies n \geq 6 \implies n = 6$$

By this argument, we can claim that $n = 6$ and $m = 36$.

4. For positive integer m, let $I(m) = \int_0^{2\pi} cos(x)cos(2x)cos(3x)...cos(mx)$. Determine all $m \leq 10$ such that $I(m)$ is non-zero. **[20]**

Solution

By de Moivre's Theorem $\cos\theta + i\sin\theta = e^{i\theta}$, we have

$$I_m = \int_0^{2\pi} \Pi_{k=1}^m \left(\frac{e^{ikx} + e^{-ikx}}{2}\right) dx = 2^{-m} \sum_{c_k=\pm 1} \int_0^{2\pi} e^{i(c_1+2c_2+\cdots+mc_m)x} dx$$

where the sum ranges over the 2^m m-tuples $(c_1, \cdots, c_m)$ with $c_k = \pm 1$ for each k. For $t \in \mathbf{Z}$

$$I = \int_0^{2\pi} e^{itx} dx = 2\pi \text{ if } t = 0 \text{ and } I = 0 \text{ otherwise}$$

Thus $I_m = 0 \iff 0 = c_1 + 2c_2 + \cdots + mc_m$ for some $c_1, \cdots, c_m \in \{1, -1\}$. If such c_k exist, then $0 = c_1 + 2c_2 + \cdots + mc_m \equiv 1 + 2 + \cdots + m = m(m+1)/2 \ (mod\ 2)$ so $m(m+1) \equiv 0 \ (mod\ 4)$, which forces $m \equiv 0$ or $3 (mod\ 4)$.

Conversely, if $m \equiv 0 (mod\ 4)$, then

$0 = (1-2-3+4) + (5-6-7+8) + \cdots + ((m-3)-(m-2)-(m-1)+m)$,

and if $m \equiv 3(mod 4)$, then

$0 = (1+2-3)+(4-5-6+7)+(8-9-10+11)+\cdots+((m-3)-(m-2)-(m-1)+m)$

Thus $I_m = 0 \iff m \equiv 0$ or $3(mod 4)$. The integers m between 1 and 10 satisfying this condition are 3, 4, 7, 8.

5. Show that $n = 2116^{2001} - 2025^{2001} - 2039^{2001} + 1948^{2001}$ is divisible by 2002. **[20]**

Solution

Since, $2116^{2001} - 2025^{2001}$ is divisible by $2116 - 2025 = 91$ and

$2039^{2001} - 1948^{2001}$ is divisible by $2039 - 1948 = 91$ so n is divisible by $91 = 13 \times 7$

Similarly $n = (2116^{2001} - 2039^{2001}) - (2025^{2001} - 1948^{2001})$ is divisible by $77 = 7 \times 11$

Also it is trivial that n is even.

Therefore n is divisible by $2 \times 7 \times 11 \times 13 = 2001$

6. Three large glasses contain a,b and c liters of water (a, b, c being positive integers). One is allowed to double the contents of a glass by pouring in water from one of the other two glasses. eg: if $a \leq b \leq c$ then from the state (a, b, c), in one move, one can go to $(2a, b - a, c)$ or $(2a, b, c - a)$ or $(a, 2b, c - b)$ but no other state. Determine the condition on a, b, c that will allow us, through a legal sequence of moves, to achieve the following: **[15+15]**

   (a) empty at least one glass

   Solution

   Claim: It is always possible to empty at least one glass.

   Initially, $0 < a \leq b \leq c$. It is enough to prove that through a sequence of moves we can get a glass with amount of water strictly less than a. Let $b = qa + r$ where q is the quotient and $r < a$ the remainder. We shall reduce amount of water in glass B to r.

   Let $q = (q_n \dots q_1 q_0)_2$ (binary representation). Perform $n + 1$ steps. If $q_i = 0$ then in $(i + 1)^{st}$ move double contents of A from C else double contents of A from B. Note that before $(i + 1)^{st}$ move, A contains exactly $2^i a$ amount of water. So amount of water poured out of B is exactly $= \sum_{i=0}^{n} q_i 2^i a = qa$. Hence, amount of water in B is now r as desired. To complete the proof, we must show that all moves involving C were valid (ie C always had enough water to double the content of a during the process).

   Total amount of water poured out of C is exactly $= \sum_{i=0}^{n} (1 - q_i) 2^i a < qa < b \leq c$. Now the proof is complete.

   (b) empty 2 glasses.

   Solution

   Two glasses can be emptied $\iff$ $(a + b + c)/d$ is a power of 2 where $d = \gcd(a, b, c)$.

   Proof: We may clearly assume $d = 1$ So no odd prime p divides each of a, b and c. This remains valid throughout the process and hence, if we are to empty two glasses, no odd prime should divide $(a + b + c)$. Thus $(a + b + c)$ must be a power of 2.

   Now let $a + b + c = 2^k$.We shall prove that we can empty two glasses by induction on k. Base case $k = 0$ is trivial. For $k > 1$, we can empty one glass for sure (part 1 of this question). So the configuration is say $(a_1, b_1, 0)$ $\quad a_1 \leq b_1$ and $a_1 + b_1 = 2^k$. Doubling a_1 we get $(2a_1, b_1 - a_1, 0)$. Now all glasses have even amount of water. Hence it is equivalent to the case $(a_1, (b_1 - a_1)/2, 0)$. Hereafter, the induction argument can be easily completed.