Math Olympics 2016 : Round 3

Math and Physics Club, IIT Bombay

Time: 30 minutes

Name: E-mail:

1. Can we use straight triominoes to tile a standard checkerboard (8×8) with one of its four corners removed? Straight triominoes refer to a unit of three unit squares, arranged end-to-end. **[35]**

Solution

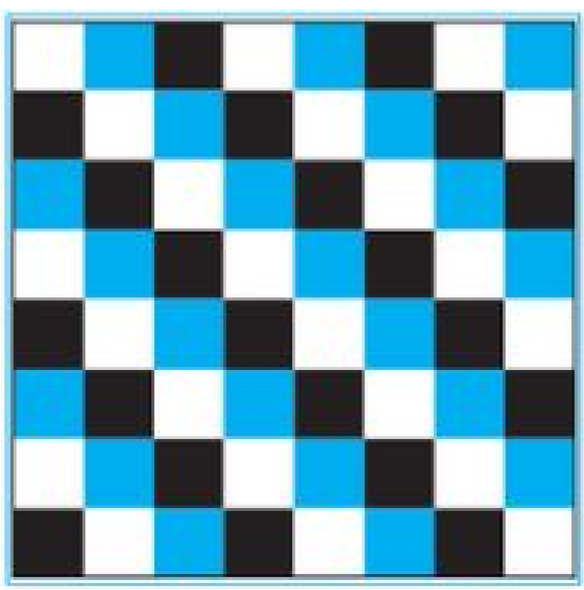
We will colour the squares of the checkerboards in an attempt to adapt the proof by contradiction of the impossibility of using dominoes to tile a standard checkerboard with opposite corners removed.

Because we are using straight triominoes rather than dominoes, we colour the squares using three colours rather than two colors as shown in the adjoining figure.

Note that thre are 21 blue squares, 21 black squares and 22 black ones in this colouring. Next, we make the crucial observation that when a straight triomino covers three squares of the checkerboard, it covers one blue, black and white square respectively.

Note that each of the three colours appears in the corner squares. Thus, without loss of generality, we may assume that we have rotated the colouring so that the missing square is coloured blue. Therefore, we assume that the remaining board contains 20 blue squares, 21 black and 22 white squares.

If we could tile this board using straight triominoes then we would use $63/3 = 21$ straight triominoes.These would cover 21 blue, 21 black and 21 while squares.This contradicts the fact that the board has 20 blue squares, 21 black and 22 white squares. Therefore we cannot tile this board with straight triominoes.



2. A positive integer is called *fancy* if it can be expressed in the form $2^{a_1} + 2^{a_2} + ... + 2^{a_{100}}$ where a_1, a_2, ..., a_{100} are non-negative integers, that are not necessarily distinct. Find the smallest positive integer n such that no multiple of n is a fancy number. **[45]**

Solution

We shall prove that $n = 2^{101} - 1$ is the one. Indeed, consider any number that is less than $n = 2^{101} - 1$ then n can be represented uniquely as -

$$n = 2^{b_1} + ... + 2^{b_k}$$
$$\text{with } 0 \leq b_1 < b_2 < ... < b_3 \text{ and } k \leq 100.$$

We prove that there is a multiple of n that is a fancy number. We can times n by 2^l for large enough l to get -

$$2^l n = 2^{l+b_1} + ... + 2^{l+b_k}.$$

Using a simple separation $2^{x+1} = 2^x + 2^x$ to separate each 2^{l+b_i} until we reach exactly 100 numbers.

Hence, 2^l is a fancy number. This means $n \geq 2^{101} - 1$.

For $n = 2^{101} - 1$, we will induct on k that $(2k+1)n$ is not a fancy number and $2kn$ is not a *fancy* number.

It's obviously true for $k = 1$. If it's true until $k = m - 1$, we need to prove that $(2m+1)n, 2mn$ are not *fancy* numbers.

Assume contradiction, which means -

$$(2m+1)(2^{101} - 1) = 1 + 2^{b_1} + ... + 2^{b_{100}}, \forall (b_i \in \mathbb{N}, b_i \geq 0)$$
$$\text{or, add } (2^{101} - 1) \text{ on both sides and divide by 2,}$$
$$(m+1)(2^{101} - 1) = 2^{100} + 2^{b_2 - 1} + 2^{b_{100} - 1} = 2^{100} + l,$$

with $l \geq 2$ it's not hard to observe that $l \in \mathbb{N}$.

Next we notice that $2^{100} = 2 + 2 + 2^2 + ... + 2^{99}$ so 2^{100} can be written as -

$$2^{100} = 2^{a_1} + 2^{a_2} + ... + 2^{a_{99}}, \forall (a_i \geq 1, 1 \leq i \leq k) for any 1 \leq k \leq 101$$

Also, note that $l = 2^{b_2 - 1} + ... + 2^{b_{100} - 1}$ so the number of digits of l in base 2 must be at most 99. From this, it follows that we can always write $2^{100} + l$ as a *fancy* number. This gives contradiction to the hypothesis since $m+1 < 2m$ so $(2m+1)n$ isn't a *fancy* number.

Now, for $2mn$ if it is a *fancy* number, or $(2^{101} - 1)2m = 2^{a_1} + 2^{a_2} + ... + 2^{a_{100}}$ with $1 \leq a_1 \leq a_2 \leq ... \leq a_{100}$ implies $(2^{101} - 1)m$ is also *fancy*, a contradiction to the induction hypothesis.

Thus, $2m$ cannot be a *fancy* number. QED