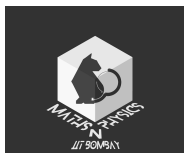


Mathathon 2019

Round 4



Maths and Physics Club, IIT Bombay

2nd October, 2019

Name:

E-mail:

Freshie/Senior

1. A sequence of positive integers $(a_n)_{n \geq 1}$ is said to be of *Fibonacci type* if it satisfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?
2. Let ABC be an acute triangle with circumcircle ω , and let H be the foot of the altitude from A to \overline{BC} . Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB at E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of $\triangle AE_1F_1$ and $\triangle AE_2F_2$ are congruent, and the line through their centers is parallel to the tangent to ω at A .
3. Let (a, b) be a lattice point, that is, $(a, b) \in \mathbb{Z}^2$. We say that (a, b) is *visible from the origin* if there does not exist any other lattice point on the line joining the origin to (a, b) . Prove that the set of lattice points in the plane visible from the origin contains arbitrarily large square gaps. That is, given any integer $k > 0$, there exist a lattice point (a, b) such that none of the lattice points

$$(a + r, b + s), \quad 0 < r \leq k, 0 < s \leq k,$$

is visible from the origin.

4. Let n be a positive integer satisfying the following property: If n dominoes are placed on a 6×6 chessboard with each domino covering exactly two unit squares, then one can always place one more domino on the board without moving any other dominoes. Determine the maximum value of n .
5. Prove that for any $n \geq 2$,

$$\sum_{p \leq n, p \text{ prime}} \frac{1}{p} > \ln \ln n - 1.$$