Mathathon 2019
Round 4

Maths and Physics Club, IIT Bombay

2^{nd} October, 2019

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Freshie/Senior

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1. A sequence of positive integers $(a_n)_{n\geq 1}$ is said to be of *Fibonacci type* if it satsfies the recursive relation $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 1$. Is it possible to partition the set of positive integers into an infinite number of Fibonacci type sequences?

2. Let ABC be an acute triangle with circumcircle ω, and let H be the foot of the altitude from A to $\overline{BC}$. Let P and Q be the points on ω with $PA = PH$ and $QA = QH$. The tangent to ω at P intersects lines AC and AB and E_1 and F_1 respectively; the tangent to ω at Q intersects lines AC and AB at E_2 and F_2 respectively. Show that the circumcircles of ΔAE_1F_1 and ΔAE_2F_2 are congruent, and the line through their centers is parallel to the tangent to ω at A.

3. Let $(a,\ b)$ be a lattice point, that is, $(a,\ b) \in \mathbb{Z}^2$. We say that $(a,\ b)$ is *visible from the origin* if there does not exist any other lattice point on the line joining the origin to $(a,\ b)$.
   Prove that the set of lattice points in the plane visible from the origin contains arbitrarily large square gaps. That is, given any integer $k > 0$, there exist a lattice point $(a,\ b)$ such that none of the lattice points
   $$(a+r,\ b+s), \quad 0 < r \leq k, 0 < s \leq k,$$
   is visible from the origin.

4. Let n be a positive integer satisfying the following property: If n dominoes are placed on a 6×6 chessboard with each domino covering exactly two unit squares, then one can always place one more domino on the board without moving any other dominoes. Determine the maximum value of n.

5. Prove that for any $n \geq 2$,
   $$\sum_{p\leq n,\ p \text{ prime}} \frac{1}{p} > \ln\ln n - 1.$$