Combinatorics

MnP Club IIT Bombay

June 2024

1 Pigeon-Hole Principle

The Pigeonhole Principle (also sometimes called the Box Principle or the Dirichlet Box Principle) simply states that if one wants to put pigeons in holes, and there are more pigeons than there are holes, then one of the holes has to contain more than one pigeon. There is also a stronger form of the principle: if the number of pigeons is more than k times the number of holes, then some hole has at least k + 1 pigeons.
Exercise: Suppose a soccer team scores at least one goal in 20 consecutive games. If it scores a total of 30 goals in those 20 games, prove that in some sequence of consecutive games it scores exactly 9 goals.
Hint: Consider the number of goals in the first i days, and also the number of goals + 9 for each of the first i days.
Example: Dirichlet's Theorem on Diophantine Approximation
If α is a real irrational number, then there are infinitely many rationals $\frac{p}{q}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.
Proof: First, we show that if α is real and if Q is some arbitrary real greater than 1, there exists integers q and p with $1 \leq q < Q$ and $gcd(p,q) = 1$ such that $|q\alpha - p| \leq \frac{1}{Q}$.
Let $N = \lceil Q \rceil$. Consider the $N+1$ numbers 0, 1, $\{\alpha\}, \cdots, \{(n-1)\alpha\}$, where $\{x\}$ denotes the fractional part of x. If we divide the unit interval into N pieces of size $\frac{1}{N}$, there must be two which are in the same interval. The difference between the two has the form $q\alpha - p$, where p and q are integers with $0 < q < N$. These are the required values of p, q.
Now, we know that for any Q, there exists such p,q with $|\alpha - \frac{p}{q}| \leq \frac{1}{Qq} \leq \frac{1}{q^2}$. Let Q be any real number exceeding $\frac{1}{\alpha - p/q}$. A second application of the above result shows us that there exists p', q' such that $|\alpha - \frac{p'}{q'}| \leq \frac{1}{Q'q'} \leq \frac{|\alpha - \frac{p}{q}|}{q'} \leq \left|\alpha - \frac{p}{q}\right|$.
By iterating this process, we obtain a sequence $(\frac{p_n}{q_n})_{n=1}^{\infty}$ of distinct rational numbers with

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \left|\alpha - \frac{p_{n-1}}{q_{n-1}}\right| < \cdots < \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

and hence we are done.
Exercise: Fifty-one points are placed in a square of side length 1. Prove that there is a circle of radius $\frac{1}{7}$ that contains three of the points. **Exercise:** Let $x_1, \cdots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, \cdots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.
Solution: FTSOC, suppose this is not the case. Without loss of generality, $\sum x_i > \sum y_i$. Define $f(k)$ to be the smallest value j such that $\sum_{i=1}^{j} x_i > \sum_{i=1}^{k} y_i$.
Note that the difference between the partial sums $\sum_{i=1}^{f(k)} x_i - \sum_{i=1}^{k} x_i$ is in the set $\{1, \cdots 92\}$ since it is not 0 and is capped by $x_j - 1$. But there are 93 partial sums, so two must be identical.

$$\sum_{i=1}^{j_1} x_i - \sum_{i=1}^{k_1} y_1 = \sum_{i=1}^{j_2} x_i - \sum_{i=1}^{k_2} y_1$$

Taking the difference,

$$\sum_{i=j_2+1}^{j_1} x_i = \sum_{i=k_2+1}^{k_1} y_1$$

With this, we are done with the proof.
Exercise: Let B be a set of more than $\frac{2^{n+1}}{/}n$ distinct points with coordinates of the form $(\pm 1, \cdots, \pm 1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.
Solution: Let S be the set of all points in n-dimensional space with coordinates of the form $(\pm 1, \cdots, \pm 1)$. For each $P \in B$, let S_P be the set of points in S which differ from P in exactly one coordinate. Since there are more than $\frac{2^{n+1}}{n}$ points in B, and each S_P has n elements, the cardinalities of the sets S_P sum to more than 2^{n+1}.
But this number is more than twice the number of points in S. By the Pigeonhole Principle, there must be a point of S in at least three of the sets, say in S_P , S_Q, S_R. But then any two of P, Q, R differ in exactly two coordinates, so P QR is an equilateral triangle.
Exercise: A sequence of m positive integers contains exactly n distinct terms. Show that if $2^n \leq m$, there exists a block of consecutive terms whose product is a perfect square.
Solution: Consider a function with the domain $1, \cdots, m$ with $f(k)$ returning a tuple of the number of times a_i appears in the first k terms of the sequence, modulo 2. If any value of $f(k)$ is $(0, \cdots, 0)$, then we are done, as this implies that in the sequence from 1 to k, each of the n terms appears an even number of times. If not, there are at most $2^n - 1$ values $f(k)$ can take, and $\geq 2^n$ values of k. Hence, $f(k)$ must be equal at two points. The block between these two values of k is our required block.
Exercise: Prove that there exists a positive integer n such that the four leftmost digits of the decimal representation of 2^n is 2024.

Exercise: Let p and q be positive integers. Show that within any sequence of pq+1 distinct real numbers, there exists either a increasing subsequence of p + 1 elements, or a decreasing subsequence of q + 1 elements.
Solution: This is a pretty standard theorem, the Erdős–Szekeres theorem. The proof can be found online, but as a hint, consider the function which maps every index to the longest increasing and longest decreasing subsequence ending at that index.
Exercise: Let S = $\{1, \cdots 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.
Solution: The answer is 217.
Let A be the subset of all multiples of 2, 3, 5 or 7. Then A has 216 members and every 5-subset has 2 members with a common factor. Hence n is at least 217.
Let P be the set consisting of 1 and all the primes ¡ 280. Define:

$$A_1 = \{241, 337, 531, 729, 1123, 1319\}$$
$$A_2 = \{237, 331, 529, 723, 1119, 1317\}$$
$$A_3 = \{231, 329, 523, 719, 1117, 1313\}$$
$$B_1 = \{229, 323, 519, 717, 1113\}$$
$$B_2 = \{223, 319, 517, 713, 1111\}$$

Note that these 6 sets are disjoint subsets of S and the members of any one set are relatively prime in pairs. But P has 60 members, the three As have 6 each, and the two Bs have 5 each, a total of 88. So any subset T of S with 217 elements must have at least 25 elements in common with their union. But 6·4 = 24 < 25, so by PHP, T must have at least 5 elements in common with one of them. Those 5 elements are the required subset of elements relatively prime in pairs.

2 Combinatorics for Graph Theory

Theorem 1. ***Cayley's Formula***
The number of trees with n vertices is n^{n-2}.

Proof:

1. **Prüfer Sequences:** One can generate a labeled tree's Prüfer sequence by iteratively removing vertices from the tree until only two vertices remain.
   Specifically, consider a labeled tree T with vertices $\{1, \cdots, n\}$. At step i, remove the leaf with the smallest label and set the i^{th} element of the Prüfer sequence to be the label of this leaf's neighbour until there are only

2 vertices remaining.
Thus, the Prüfer sequence has length n-2. Each vertex i occurs in the sequence $d_i - 1$ times (if a vertex is removed from the graph at some step, all its neighbours but one must have been removed before it while they are leaf nodes, hence it gets added to the Prüfer sequence once for each neighbour but one. If the vertex is one of the two which does not get removed, all its neighbours but one is removed. The same result holds.).
From the Prüfer sequence, one can generate a tree using the following algorithm: first, assign each vertex its degree (the number of times it appears in the sequence + 1). Then, for each element in the sequence, find the lowest indexed vertex which has degree 1 and add an edge between that index and the element of the sequence. Decrement the degrees of these two vertices. At the end, two vertices will remain with degree 1. Add an edge between these two.
This shows that each Prüfer sequence corresponds to a unique labeled tree, thus there are n^{n-2} such trees.

2. **Induction:** We will prove a stronger claim: Let $\{v_1, \cdots v_n\}$ be given points and $\{d_1, \cdots d_n\}$ numbers such that $\sum_{i=1}^{n} d_i = 2n-2$, $d_i \geq 1$. The number of labeled trees on $\{v_1, \cdots v_n\}$ in which v_i has degree d_i is

$$\frac{(n-1)!}{(d_1-1)! \cdots (d_n-1)!}$$

Let the number of such trees be denoted by $T(d_1, \cdots, d_n)$. We prove the required result by induction on n following by a simple application of the multinomial principle.
For the base case, if $n = 2$, then $d_1 = d_2 = 1$ and it is easy to observe the induction hypothesis holds.
For a particular sequence of d_i and a particular value of n, at least one of the d_i's must be 1. WLOG, let $d_1 = 1$. Now, we have n-1 choices. We choose $j \neq 1$ and join vertices j and vertex 1. Now we have $T(d_2, \cdots d_j - 1, d_{j+1}, \cdots d_n)$ ways to pick a tree with the remaining vertices, assuming v_j now has degree $d_j - 1$.
Now, $T(1, d_2 \cdots d_n) = T(d_2 - 1, \cdots d_n) + T(d_2, d_3 - 1, \cdots d_n) \cdots T(d_2, \cdots d_n - 1)$. By applying the induction hypothesis,

$$
\begin{aligned}
T(1, d_2, \cdots, d_n) = & \frac{(n-3)!}{(d_2-2)! \cdots ((d_n-1)!)} + \frac{(n-3)!}{(d_2-1)!(d_3-2)! \cdots ((d_n-1)!)} \\
& \cdots + \frac{(n-3)!}{(d_2-1)! \cdots ((d_n-2)!)} \\
= & \frac{(n-3)!}{(d_2-1)! \cdots ((d_n-1)!)} \cdot ((d_1-1) + \cdots + (d_n-1)) \\
= & \frac{(n-3)!}{(d_2-1)! \cdots ((d_n-1)!)} \cdot (n-1) \\
= & \frac{(n-2)!}{(d_2-1)! \cdots ((d_n-1)!)}
\end{aligned}
$$

Note that above, if $d_j = 1$, we technically do have a term of $(d_j - 2)!$ in the denominator, but practically, since we interpret that as $\frac{d_j-1}{(d_j-1)!}$, there is a factor of 0 in the numerator in all such cases, so the above induction process works.

Now, apply the multinomial theorem to get Cayley's result.

Exercise: Let $T_1, \cdots T_r$ be trees on disjoint sets of points and V be the union of vertices in all T_i's. What is the number of trees on V cointaining all T_i's as subtrees?

Solution: $(|T_1| + \cdots + |T_r|)^{r-2}$

Let us for a moment consider each subtree as a vertex in a larger tree with r vertices. If the degrees of each vertex i is d_i, we know from the previous question there are $\frac{(r-2)!}{(d_2-1)! \cdots ((d_r-1)!)}$ such trees. Now, within each subtree, we can randomly choose any of $|T_i|$ vertices to be connected to any of the incoming or outgoing edges, thus we have $|T_i|^{d_i}$ options.

Exercise: What is the number of trees on n points with n-l endpoints?

Solution: Using the Prufer sequence argument, we get the condition that n+l-2 is the sum of the degrees of the remaining l vertices, all of which have degrees greater than 1. Now, for each choice of degrees d_i's of the remaining l vertices, we have a factor of $\frac{(n-2)!}{\Pi(d_i-1)!}$. We see we can get this factor by considering the coefficient of n-2 in $\binom{n}{l}(n-2)! \left(\sum_{i=2}^{\infty} \frac{x^i}{(i-1)!} \right)^l$, which equals the coeff of n-l-2 in $\binom{n}{l}(n-2)! \left(\sum_{i=1}^{\infty} \frac{x^i}{i!} \right)^l = \binom{n}{l}(n-2)!(e^x - 1)^l$.

Exercise: Let G be a directed graph without cycles. Let A be the point-edge incidence matrix of G, given by

$$
a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the head of} e_j. \\ -1 & \text{if } v_i \text{ is the tail of} e_j. \\ 0 & \text{otherwise} \end{cases}
$$

Remove any row from A and let A_0 be the remaining matrix. Prove that the number of spanning trees of G is $det(A_0 \cdot A_0^T)$. Derive the Cayley formula from

this.
Solution: We first consider how each element of $A \cdot A^T$ looks and note that $A_0 \cdot A_0^T$ is obtained by deleting the ith row and ith column from $A \cdot A^T$. Its diagonal entries are the degrees (in-degree + out-degree) of all the vertices, and its non-diagonal entries are $a_ij = -1$ if there is an edge between vertices i and j in any direction, and 0 otherwise[1]. WLOG[2], we now eliminate the first row and first column from $A \cdot A^T$ to get an $A_0 \cdot A_0^T$.
By the Cauchy-Binet Formula, the determinant of the product of two rectangular matrices A, B of sizes $a \times b$ and $b \times a$, with $b \geq a$, Let $[a]$ refer to the set $\{1, \cdots a\}$ and $\binom{[b]}{a}$ to the set of subsets of $[b]$ of size a. For $S \in \binom{[b]}{a}$, let $A_{[a],S}$ be the $a \times a$ matrix whose columns are the columns of A with indices from S, and similarly for B.

$$det(AB) = \sum_{S \in \binom{[b]}{a}} det(A_{[a],S}) \cdot det(B_{S,[a]})$$

In our case, since B=A^T, we can further write,

$$det(A \cdot A_T) = \sum_{S \in \binom{[b]}{a}} det(A_{[a],S})^2$$

In our particular case, a=m=n-1, b is the number of edges. Every selection of a elements corresponds to a selection n-1 edges from the set of edges. We claim that $det(A_{[a],S})^2$ is 1 if these edges form a spanning tree and 0 otherwise. The second part is easy. Since the graph is cycle-free, if these n-1 edges do not span all the vertices, there are at least two components. At least one component does not contain the vertex we dropped from A. If we sum up all the rows belonging to that component, we get a row of zeroes, meaning the determinant is 0.
For the other direction, we proceed using induction. If n=2, the number of spanning trees is 1, which is the square of 1 or -1, which are the only possible determinants of the 1×1 matrix $A_0 \times A_0^T$. Suppose the induction hypothesis is true for graphs of size n-1. Let i be the dropped vertex and $j \neq i$ be a leaf. Let k be the neighbour of j. We flip rows and columns until (k,j) is the last column and j is the last row. Then, if we expand the determinant along the last row, we get that its plus or minus of the determinant of the other m-2 rows. But since removing this leaf means that the overall property of being a tree does not change, this determinant is ± 1 by induction.
Hence, we have proved the claim in the first part of the question. For the second part, simply take the directed graph on n vertices with an edge between i and j if i < j. Every subset of edges here which creates a spanning tree can be transformed into a tree in an undirected graph on the same vertices, and every tree on a labeled undirected graph is equivalent to one spanning tree of this graph. The sum of out-degree and in-degree of every vertex is (n-1) and $A \times A^T$

[1]This matrix is called the Laplacian matrix of A. Any row of A sums up to 0, since its elements are $deg(v_i)$ and -1 $deg(v_i)$ times. Thus, A itself is singular

[2]Since we could simply have permuted the rows of A earlier, there is no loss of generality.

is a symmetric graph with (n-1) $A_0 \times A_0^T$ has the same structure, but n-1 rows and columns. We need to prove that this determinant is n^{n-2}. We start off by setting the first column as the sum of all columns, so that each entry in it is +1. Then we subtract the first row from every row. It is easy to observe now that the determinant is n^{n-2} as required.
Exercise: What is the number of trees on $\{v_1, \cdots, v_n, w_1 \cdots w_m\}$ where each edge joins a v_i to a w_j?
Solution: We use the solution to the previous question. The number of spanning trees is the determinant of a matrix of the form

$$\begin{bmatrix} m & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & m & \cdots & 0 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m & -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 & n & 0 & \cdots & 0 \\ -1 & -1 & \cdots & -1 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & n \end{bmatrix}$$

We can simplify this by row and column operations to:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ 0 & m & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & -n & n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -n & 0 & \cdots & n \end{bmatrix}$$

This determinant is $m^{n-1}n^{m-1}$.
Exercise: How many k-element matchings are possible for the bipartite graph with partitions $\{v_1, \cdots v_n\}$ and $\{v_1, \cdots v_n\}$ if there exists an edge between u_i and v_j if i < j?
Solution: Note that this is equivalent to the problem of dividing n into n-k partitions - n-2k of size 1 and k classes of size 2. For every class of size 2 containing (a,b), have an edge between $u_{min(a,b)}$ and $v_{max(a,b)}$. This means that the final solution is the number of ways of partitioning n into n-k equivalence classes (note that all classes can only have sizes 1 or 2 because of the PHP). The number of such classes is the Stirling number[3] S(n,n-k).
Exercise: Iff G is a weakly-connected directed graph with indegree of each vertex equal to its out-degree, show that G has an Eulerian circuit.
Solution: One direction is trivial. If G has an Eulerian circuit, keep removing

[3] Of the second kind

simple cycles from the circuit until you are left with a graph with no edges. Since simple cycles satisfy the property of each vertex on the cycle having indegree 1 and outdegree 1, by induction, the indegree and outdegree of each vertex is equal.

If every vertex v has $indeg(v) = outdeg(v)$, the first observation is that for any vertex v, there must be a path starting from v that comes back to v.

Start from v, and chose any outgoing edge of v, say (v, u). Since $indeg(u) = outdeg(u)$ we can pick some outgoing edge of u and continue visiting edges. Each time we pick an edge, we can remove it from further consideration. At each vertex other than v, at the time we visit an entering edge, there must be an outgoing edge left unvisited, since $indeg = outdeg$ for all vertices. The only vertex for which there may not be an unvisited outgoing edge is v—because we started the cycle by visiting one of v's outgoing edges. Since there's always a leaving edge we can visit for any vertex other than v and there are a finite number of edges, eventually the cycle must return to v, thus proving the claim.

Now, pick a random vertex v and find a cycle C that comes back to v. Delete all the edges on C from G. Each vertex in the new G still has $indeg(v) = outdeg(v)$, so we pick a vertex v_0 on C that has edges incident (such a vertex must exist, since G is weakly connected) and repeat. Overall we find a cycle C, then another cycle C_0 that has (at least) a common vertex with C, and so on. Finally, we combine the cycles by constructing one that goes over C until it finds a branching cycle, enters that branching cycle and repeats the above process and then returns to the parent cycle and completes it.

Exercise: Let $G_{k,n}$ $(k \geq 2)$ be defined as follows. The points of $G_{k,n}$ are the vectors of dimension k composed from $1, \cdots n$. The vectors $(a_1, \cdots, a_k)$ and $(b_1, \cdots, b_k)$ are connected by a directed edge if $a_2 = b_1, \cdots a_k = b_{k-1}$. Show that $G_{k,n}$ is Eulerian and Hamiltonian.

Solution: Each vertex has in-degree n and out-degree n. Hence, by the above result, the graph is Eulerian.

We prove the second part by induction. If k=2, the result is trivial for any n. For a higher k, we think about $G_{k-1,n}$. In that, if we have an edge between $(a_1, \cdots, a_{k-1})$ and $(b_1, \cdots, b_{k-1})$, then we define an isomorphism from the set of edges in $G_{k-1,n}$ to the set of vertices in $G_{k,n}$ as mapping an edge between the aforementioned edge to $(a_1, \cdots, a_{k-1}, b_k)$. An Eulerian tour of $G_{k-1,n}$ covers each edge once, and if two edges coincided at a vertex in $G_{k-1,n}$, we see their corresponding vertices are adjacent in $G_{k,n}$. Hence, the Eulerian tour on $G_{k-1,n}$ is the Hamiltonian tour on $G_{k,n}$.

Exercise: Show that we can associate 0 or 1 with each point of a cycle of length 2^k such that each arc of length k gives a different unique sequence of 0s and 1s.

Solution: Think about the above question, with n=2 and the set {0,1} instead of {1,2}. A Hamiltonian path between vertices of length k can be found. Starting from an arbitrary vertex, we stitch together adjacent vertices along this Hamiltonian path, creating the required 2^k cycle.

Exercise: Prove that every edge e of a 3-regular graph is contained in an even number of Hamiltonian circuits.

Solution: We prove this by induction again. If the graph has 4 vertices, it is the complete 4-graph and each edge is contained in two cycles. Suppose G has more than 4 vertices.

Let $(x, y), (x, u_1), (x, u_2), (y, v_1), (y, v_2)$ be edges in G. Remove x and y and join u_1 to v_1, u_2 to v_2 to get G'. Remove x and y and join u_1 to v_2, u_2 to v_1 to get G".

There are four kinds of Hamiltonian circuits containing xy - u_1xyv_1, u_1xyv_2, u_2xyv_1, u_2xyv_2. Let their numbers be !, h_2, h_3, h_4. Also, the circuits which do not contain xy are of the form $\cdots u_1xu_2 \cdots v_1yv_2$ or $\cdots u_1xu_2 \cdots v_2yv_1$. Let these be h_5 and h_6 in number.

The h_1 Hamiltonian circuits above are in bijetions with those Hamiltonian circuits of G' containng u_1v_1 but not u_2v_2. Similarly, h_2, h_3 and h_4 are equal to the number of Hamiltonian Circuits of G" and G' containing u_1v_2, u_2v_1 and u_2v_2 respectively but not the other newly added edge. The h_5 circuits correspond to those in G' of the form $\cdots u_1v_1 \cdots u_2v_2 \cdots$. Similarly, h_6 is the number of circuits in G" of the form $\cdots u_1v_2 \cdots u_2v_1 \cdots$.

The Hamiltonian circuits of G' not considered so far are those of the form $\cdots u_1v_1 \cdots v_2u_2 \cdots$ and those not containing the new edges. Let them be h_7 and h_8 in number.

The Hamiltonian circuits of G" not considered so far are those of the form $\cdots u_1v_2 \cdots v_1u_2 \cdots$ and those not containing the new edges. Obviously, these are also h_7 and h_8 in number.

Now, the number of Hamiltonian circuits of G is $h_1 + h_2 + h_3 + h_4 + h_5 + h_6 \equiv (h_1 + h_4 + h_5 + h_7 + h_8) + (h_2 + h_3 + h_6 + h_7 + h_8) \equiv 0(mod 2)$, since we have in the brackets the number of Hamiltonian circuits of G' and G" respectively.

3 Important Results of Ramsey Theory

Ramsey's Theorem, extended beyond a complete 6-graph, implies that if all edges of a complete n-graph are colored by one out of a k-tuple of colors $c_1, c_2 \cdots c_k$, there exists a finite lower bound for n, $R(n_1, n_2, \cdots n_k)$, above which the n-graph has a clique of size n_i with all edges of color c_i for some i $(1 \leq i \leq k)$.

Proof:

This result can be proved by induction on k, and on $\sum_1^k n_i$ for a particular value of k. The base case is $k = 2, n_1 = n_2 = 1$. Obviously, any 1-subgraph is trivially monochromatic in either color because it has no edges, and the base case holds. Now suppose $R(n_1 - 1, n_2)$, $R(n_1, n_2 - 1)$ are both finite. If we construct a graph with $R(n_1 - 1, n_2) + R(n_1, n_2 - 1)$ vertices, and pick a random vertex, and suppose there are D_1 vertices to which it is connected by c_1, and D_2 vertices to which it is connected by c_2. $D_1 + D_2 + 1 = R(n_1 - 1, n_2) + R(n_1, n_2 - 1)$, so it cannot be the case that $D_1 < R(n_1 - 1, n_2)$ and $D_2 < R(n_1, n_2 - 1)$ simultaneously. Then, WLOG, assume $D1 \geq R(n_1 - 1, n_2)$. Within the induced subgraph created by the D_1 vertices, there exists either a c_1-clique of size $n1 - 1$ or a c_2-clique of size n_2. In the second case, we are done. In the first case,

since each edge from the originally chosen vertex to any of these D_1 vertices is of color c_i, we add the original vertex to the c_1-clique and we are done.
To proceed further, we now induct on k. Suppose that all Ramsey numbers are finite for up to k-1 colors. We claim that $R(n_1, \cdots, n_k) \leq R(n_1, \cdots, n_{k2}, R(n_{k1}, n_k))$. For the proof, consider a complete graph with $R(n_1, \cdots, n_{k2}, R(n_{k1}, n_k))$ vertices. If the first k-2 colors do not satisfy the Ramsey property, then there must be clique of color $c^{'}_{k-1}$ with a clique of size $R(n_{k1}, n_k)$. Now, simply replace all edges of color $c^{'}_{k-1}$ with either color c_{k-1} or c_k, and the required result follows.

Theorem: (Schur) There exists a lower bound n such that the set $1, 2 \cdots n$ can be colored with k colors such that there does not exist a monochromatic triple x, y, z with $x + y = z$.
Proof:
Let n be greater than $R(3, \cdots 3)$, where 3 occurs k times. Consider a complete graph with the first n natural numbers as vertices and the edge between two numbers i and j having the same color as $(i - j)$. By the Ramsey theorem, there exists a monochromatic triangle, between vertices i, j, k (say). WLOG, let $i < j < k$. Simply set $x = (j - i), y = (k - j), z = (k - i)$.
Theorem: (Van der Waerden) For every choice of natural numbers p, k there exists a number $W(p, k)$ such that for $n \geq W(p, k)$, and for any coloration of $1, \cdots n$ using k colors, there exists a p-term monochromatic arithmetic progression.
Proof:
The van der Waerden theorem can be proven by proving a slightly stronger result using induction. First, let us define a l-equivalence class for a particular number i as the subset of $[0, l]^m$ where l occurs in the i right-most positions of any set in the equivalence class and nowhere else. We can prove by induction that for any k, there exists a number $N(l, m, k)$ so that for any coloring function $f : [1, N(l, m, k)] \to [1, k]$ (i.e a function from the set of numbers $\leq N(l, m, k)$ to the set of colors $[1, k]$), there exist positive integers $a, d_1, \cdots d_m$ such that $f(a + \sum_{i=1}^{m} n_i \cdot d_i)$ is constant for any m-tuple $(n_1, \cdots n_m)$ from each k-equivalence class.
Obviously, now, if this proposition holds when m=1 for all l and k, van der Waerden's theorem is proved since $C(a + id)$ is constant for any i in the range $[0, l - 1]$.
Proof by induction:

1. Claim: $P(l, m) \implies P(l, m + 1)$
   For a fixed k, let $M = N(l, m, k), M' = N(l, m, k^M)$ and suppose that $f_1 : [1, MM'] \to [1, r]$. Define $f_2' : [1, M'] \to [1, k^M]$ so that $f_2(a) = f_2(b)$ iff $f_1(aM-j) = f_1(bM-j)$ for all $0 \leq j < M$. By the inductive hypothesis, there exists p and q such that $f_2(p + nq)$ is a constant for $n \in [0, l]$. Let $I = [pM - (M - 1), pM]$. Since $P(l, m)$ applies on this interval, there exist $a, d_1, \cdots d_m$ satisfying the condition of $P(l, m)$. Set $d_i' = d_{i-1}$ for $2 \leq i \leq m + 1$ and $d_1' = qM$. Then $P(l, m + 1)$ holds.

2. Claim: $P(l, m)$ holding for all $m \geq 1 \implies P(l + 1, 1)$

For a fixed k, let $f : [1, N(l,k,k)] \to [1,k]$ be given. Then there exists $a, d_1, \cdots d_k$ such that for $x \in [0,l]$, $a + \sum_{i=1}^{k} x_i d_i$ is bounded above by $N(l,k,k)$ and $f(a + \sum_{i=1}^{k} x_i d_i)$ is constant on l-equivalence classes. By the Pigeonhole Principle, there exist $1 \leq u \leq v \leq k+1$ such that

$$f\left(a + \sum_{i=u}^{k} ld_i\right) = f\left(a + \sum_{i=v}^{k} ld_i\right)$$

Therefore,

$$f\left(\left(a + \sum_{i=v}^{k} ld_i\right) + x\left(\sum_{i=u}^{v-1} d_i\right)\right)$$

is constant for $x \in [0,l]$. This proves P(l+1,1).

Exercise: There are 9 points in space such that no four of them are coplanar. n edges are drawn between the points to form a simple graph ($n \leq 36$). The edges are either painted red or blue. Find the smallest possible value of n such that how the n edges are drawn and painted, there must exist a red or blue triangle.

Solution: The smallest possible value of n is 33. Since a complete graph with 9 vertices has at most $\binom{9}{2} = 36$ edges, there will be 3 edges missing. For each missing edge, we remove one of its endpoint and also its connection with other vertices. This results in a complete subgraph with at least 6 vertices. As the edges of the subgraph are painted in red or blue, by the Ramsey theorem, there is a red triangle or a blue triangle.

To prove the minimal property of 33, we have to construct a graph with 32 edges such that it has no red triangle or blue triangle. From the first part of the proof, we see that there should not be a complete subgraph of 6 vertices if no red or blue triangle appears. To avoid such a complete subgraph of 6 vertices, the 4 missing edges should have distinct vertices. Suppose the vertices are A, B, C, D, a, b, c, d, X, then the missing edges should be Aa, Bb, Cc, Dd (Aa means an edge connecting A and a). By considering how any pentagon without a monochromatic red or blue triangle must look like, we can arrive at the following conclusion:

Blue edges: XA, AB, BC, CD, DX, Xa, aB, Ab, bC, Bc, cD, Cd, dX, ac, ad, bd.

Red edges: XB, BD, DA, AC, CX, Xb, bD, Bd, dA, Da, aC, Ac, cX, ab, bc, cd.

Missing edges: Aa, Bb, Cc, Dd.

We can check that this graph has no monochromatic triangle.

Exercise: 2-color the edges of a complete graph with more than 2 vertices. Prove that there is a Hamiltonian circuit which is either monochromatic, or consists of two monochromatic arcs.

Solution: We proceed by induction on n. For n=3, the hypothesis is easily proven. Suppose $n > 3$. We remove one point x. By the induction hypothesis, we find that there is a n-1 circuit which is either monochromatic or has two monochromatic arcs. In the first case, replace the edge between any two points in the Hamiltonian circuit with edges to x, and at worst case we have a circuit which has two monochromatic arcs. In the second case, if there are any two points which are adjacent on the circuit and which form a monochromatic clique with x, we similarly insert edges to x instead of the original edges. If not, the color of the edges from x to each one of two adjacent points on the path must be different. That is, if x is connected to one point by a Blue edge, x is connected to the other by Red. At the point of intersection of a blue and red edge, one of two possibilities can occur - either there are two triangles xBuBv'Rx and xRv'RvBx or xRuBv'Bx and xBv'RvRx. In the first case, delete uv' and add ux and xv'. In the second case, delete v'v and add v'x and xv. (In the sentences above, let ab denote the edge between points a and b, let aBb denote a Blue edge between a and b and let aRb denote a Red edge between a and b).

4 Double Counting Arguments (Optional)

Theorem: (Handshake Lemma) The sum of degrees of all vertices in an undirected graph is twice the number of edges.
Proof: Left as an exercise to the readers.

Theorem: (Reiman, 1958) If a graph with n vertices and E edges has no subgraph which is a 4-cycle,

$$|E| \leq \frac{n}{4}(1 + \sqrt{(}4n - 3))$$

Proof: Let us consider the number of triples of vertices u, v, w (unordered with respect to v and w), such that there exist edges between u and v and u and w. To avoid 4-cycles, each pair of vertices v, w can have at most one corresponding u. Each vertex u corresponds to $\binom{d}{2}$ such pairs, where d is the degree of the vertex. Thus,

$$\sum_{i=1}^{n} \binom{d_i}{2} \leq \binom{n}{2}$$

$$\sum_{i=1}^{n} d_i^2 \leq n(n-1) + \sum_{i=1}^{n} d_i$$

By the Cauchy–Schwarz inequality, $\sum_{i=1}^{n} d_i^2 \geq \frac{(\sum_{i=1}^{n} d_i)^2}{n}$, therefore

$$(\sum_{i=1}^{n} d_i)^2 \leq n^2(n-1) + n\sum_{i=1}^{n} d_i$$

By the Handshake Lemma, $\sum_{i=1}^{n} d_i = 2|E|$. We thus get,

$$4(|E|)^2 - 2n|E| - n^2(n-1) \leq 0$$

Solving this in terms of $|E|$ gives the required result.

Theorem: (On Turán's numbers) Define the Turán number $T(n, k, l)$ as the smallest number of l-element subset of an n-element set X such that every k-element subset of X contains at least one of these sets.

$$T(n, k, l) \geq \frac{\binom{n}{l}}{\binom{k}{l}}$$

Proof: Let A be a set of l-element sets satisfying the above condition. Construct a boolean matrix M, each of whose rows represents a set P in A and each of whose column represents a k-element subset Q of X. Let an entry $m_{P,Q}$ be 1 iff $P \subseteq Q$. Let r_P be the number of ones in the P-th row and c_Q be the number of ones in the Q-th column. r_P is the number of k-element supersets of a l-ement subset, so $r_A = \binom{n-l}{k-l}$. Each column corresponds to a k-element subset of X, which contains at least one of the l-element sets in A, hence $\sum c_Q \geq \binom{n}{k}$.

$$|A|\binom{n-l}{k-l} = \sum_{P \in A} r_P = \sum_{Q \subseteq X} c_Q \geq \binom{n}{k}$$

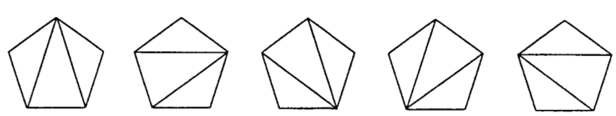
Thus,

$$T(n, k, l) \geq \frac{\binom{n}{k}}{\binom{n-l}{k-l}} = \frac{\binom{n}{l}}{\binom{k}{l}}$$

5 Catalan Numbers (Optional)

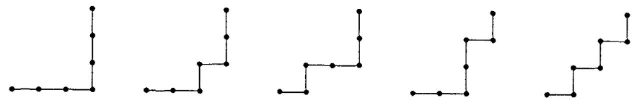
It is well known that the Catalan numbers, denoted by $C_n = \frac{\binom{2n}{n}}{n+1}$, are given by the recursion $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$. A simple proof for this fact can be found in most books on combinatorics or even Wikipedia.
We now prove to you that the below counting problems are equivalent to the problem of Catalan numbers.

1. Division of a (n+2)-gon into n triangles by non-intersecting diagonals.



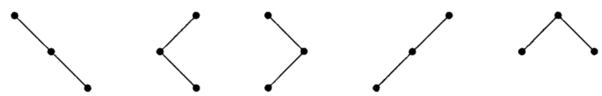
   Proof: Pretty trivial: consider the two polygons left after choosing one diagonal and you arrive at the recursive definition.

2. Lattice paths from (0,0) to (n,n) with Right or Up steps, never rising above the main diagonal.



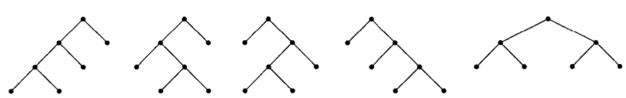
Proof: The total number of lattice paths is $\binom{2n}{n}$. Out of these, consider all bad paths that rise above the main diagonal. From the first point that such a path rises above the diagonal, flip every Up to Right and Right to Up. Initially, there was one more Up move than Right move, in order to cross the diagonal. Now, there will be two more Up moves (n+1) than Right moves. Because every path from (0, 0) to (n-1, n+1) has to touch the line above the main diagonal, and because the reflection process is reversible, the reflection is therefore a bijection between bad paths and all paths to (n-1, n+1). Hence there are $\binom{2n}{n+1}$ bad paths. Subtracting the bad paths from the good, we get the closed form formula for the n-th Catalan number.

3. Binary trees with n vertices.



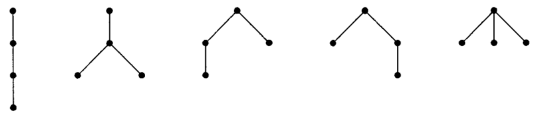
Proof: Consider the number of vertices to the left and the right of the root vertex. They are of the form i, n-1-i for i varying from 0 to n-1. Obviously, the sub-trees are also binary trees. It is hence obvious how these binary trees satisfy the Catalan recursion.

4. Binary trees with each vertex having 0 or 2 children, and 2n+1 vertices in total.



Proof: Take away all the edges (observe that they are always n+1 in number under the given constraints), and it is easy to observe that we are left with the set of all binary trees with n vertices. Alternatively, do a DFS of the graph, recording (when you go to the left and) when you take the path along the right. This sets up a bijection with the number of regular bracket sequences of length 2n.

5. Plane trees with n+1 vertices.

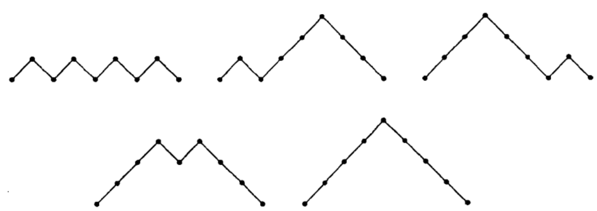


Proof: Consider the Newick format representation of the tree. Put simply, for each vertex, brackets are put around the subtree with that vertex as the root, and this is recursively done. For instance, the five graphs above correspond to (((()))), ((()())), ((())()), (()(())) and (()()()). Remove the outermost bracket, corresponding to the root vertex, and you are left with a regular bracket sequence of length 2n. We have seen above that these regular bracket sequences are equivalent to the Catalan numbers.

6. Paths from (0,0) to (2n,0) with steps (1,1) and (1,-1) only (Dyck paths) never falling below the x-axis.

Proof: Take every step (1,1) to be a left bracket and (1,-1) to be a right bracket. This question is then equivalent to finding the number of regular bracket sequences of length 2n.

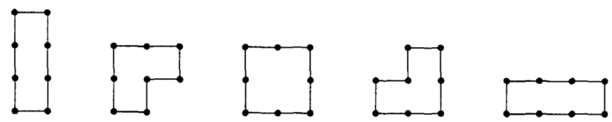
7. Dyck paths from (0, 0) to (2n+2, 0) with no peaks at height 2.



Proof: Consider the graphs equivalent to these Dyck path. It is easy to observe that these are all the plane graphs with n+2 vertices such that there are no leaves at a height two below the root. Let this number be f(n). If we split the graph at the root to get two subtrees of size i+1 and n-i+2, we have f(i-1) and f(n-i) choices for the two subtrees. Thus, $f(n) = \sum_{i=1}^{n} f(i-1)f(n-i)$, which is the Catalan recursion. Now, all that remains is to check that the base case (n=0) for the given question is C_0.

8. Lattice paths with n+1 steps each, starting at (0,0) using steps of (0,1) or (1,0), ending at the same point, and only intersecting at the beginning or

end.

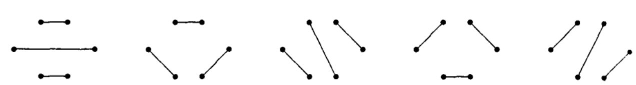


Proof: Divide the area bounded by the two paths into unit squares, with, say, k columns. Let a_i be the number of squares in the i$^{\text{th}}$ column. Let b_i be the number of rows common to the (i-1)$^{\text{th}}$ and (i)$^{\text{th}}$ columns, with $b_1 = 1$. Define a sequence of k numbers x_i and y_i $x_i = a_i - b_{i-1} + 1$, $y_i = a_i - b_i + 1$. We alternate between x_i ('s and y_i)'s. We need to show that this forms a regular bracket sequence. Note that $\sum_{i=1}^{j}(x_i - y_i) = b_j - 1 \geq 0$. Also, $\sum_{i=1}^{k} x_i = \sum_{i=1}^{k} y_i = \sum_{i=1}^{k} a_i - \sum_{i=0}^{k-1} b_i + k$. $\sum_{i=1}^{k} a_i - \sum_{i=0}^{k-1} b_i$ equals the number of UP moves - 1 and k is the number of RIGHT moves. Hence $\sum_{i=1}^{k} a_i - \sum_{i=0}^{k-1} b_i + k =$ n. Given a random regular bracket sequence of length n, since we know $b_0 = 1$, it is easy to see how the values of a_i and b_i can be constructed from the sequence, giving us a bijection between regular sequences of length n and these pairs of lattice walks of length n+1.

9. Pairs of lattice paths with n-1 steps each, starting at (0,0) and using steps of (1,0) or (0,1), such that one never rises above the other.

Proof: Note that in the paths in the last question we considered, one player always moves UP in his first move and RIGHT in his last, and the other player moves RIGHT in his first and UP in his last. Since these are fixed, we can delete these moves from the paths, thus getting a bijection between the pairs of paths in the previous question and the pairs of paths in this question.

10. n non-intersecting chords joining 2n points on the circumference of a circle.



Proof: We note that, when we choose the first chord, we need to leave an even number of vertices on either side, ranging from 0 to 2n-2. We thus get the recursion $f(n) = \sum_{i=0}^{n-1} f(i) \cdot f(n-1-i)$, if f(n) is the number of ways of forming n non-intersecting chords joining 2n points on the circumference of a circle. Thus, we see that f() follows the Catalan recursion and all that remains to be proven is the base case.

11. Ways of connecting 2n points on a line by n non-intersecting arcs containing two points each and lying above the points.

Proof: Simply make the points now lie on the circumference of a circle. If two arcs did not intersect initially, the points connected by one arc will not lie on opposite sides of the chord corresponding to the other arc. Thus, this is equivalent to the last question considered.

12. Ways of drawing in the plane n+1 points lying on a horizontal line L and n arcs connecting them such that the arcs do not pass below L, the graph thus formed is a tree, no two arcs intersect in their interior and at every vertex, all arcs exit in the same direction - either left or right.

Proof: We read the vertices from left to right. Add a (for every arc exiting to the right and a) for every arc entering from the right. Since every entering arc must have begun somewhere and every arc contributes one (and one), this forms a regular bracket sequence.
We further need to show every regular bracket expression satisfies the above condition. This is obvious if we translate the regular bracket expression to the corresponding graph, and then consider this sequence of arcs to be a DFS on the graph. When the DFS algorithm jumps back from a leaf, we add that leaf as a point to the line and connect that point to the last point along its path to the root which still has unvisited branches. If we reach a point which is not a leaf, we simply add it to the line.

13. Sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$:

111   112   113   122   123

Proof: For each lattice path from (0,0) to (n,n) not crossing the diagonal, let $a_i - 1$ be the maximum height reached by the lattice path when $x = i - 1$. This gives us a bijection between lattice paths and these numbers.

14. Sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers with $1 \leq a_i \leq 2i$.

12 13 14 23 24

Consider the sequence $b_i = a_i - i$. Let b_i be the maximum height achieved by a lattice path from (0,0) to (n,n) not crossing the diagonal at x=i. This gives us a bijection from such lattice paths to these numbers.

15. Sequences $a_1, \cdots a_n$ of integers with $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$.

000 001 010 011 012

Proof: Consider the sequence $i - a_i$. This is a non-decreasing sequence with $1 \leq a_i \leq i$.

16. Sequences of n-1 integers a_i such that $a_i \leq 1$ and all partial sums are non-negative.

$0, 0 \quad 0, 1 \quad 1, -1 \quad 1, 0 \quad 1, 1$

Consider the new sequence $b_i = \sum_{j=1}^{i} 1 - a_j$. Since $\sum_{j=1}^{i} a_j \geq 0$, $\sum_{j=1}^{i} 1 - a_j \leq i$ and it is also non-decreasing, since b_i is the sequence of partial sums of a non-negative sequence.

17. Sequences of n integers such that $a_i \geq -1$, all partial sums are non-negative, and the total sum is equal to 0.

$0, 0, 0 \quad 0, 1, -1 \quad 1, 0, -1 \quad 1, -1, 0 \quad 2, -1, -1$

Let the sequence b_i consist of the partial sums of the sequence $a_i + 1$. b_i is a non-decreasing sequence and $b_i \geq i$, with $b_n = n$. Let b_i denote the max distance to the right achieved when $y = i$ in a lattice walk where the aim is to not go below the main diagonal from (0,0) to (n,n). This gives a bijection between the sequences in the question and the lattice walks described.

18. Permutations $a_1, a_2 \cdots a_n$ of n with longest decreasing subsequence of length at most 2. These are called 321 avoiding permutations.

123 213 132 312 231

Proof: In a permutation, we identify the indices of all the numbers which are greater than every number before them (the first index is also in this set). Let the set of indices be $< a_i >$ and the corresponding set of numbers be $< b_i >$. Note that all the numbers $\leq n$ which are not the set $< b_i >$ have to be filled in ascending order in the indices which are not in $< a_i >$. If not, and if two of them are ordered in decreasing order, since the larger of the two itself is not the maximum of all the numbers until it, we have a decreasing sequence of length 3. Thus, $< b_i >$ and $< a_i >$ uniquely determine this permutation.
We take a lattice walk from (0,0) to (n,n) as follows: If $y + 1$ is not a_i, then we go UP. Else, we go RIGHT until $x = b_i$ and then go UP once. It is easy to observe that $a_i - 1 > b_i$ is not possible (as this would imply the largest number in the first i numbers is less than i-1, which is not possible for a permutation), hence we never cross the main diagonal. Furthermore, a lattice walk uniquely determines $< a_i >$ by the y-coordinates at which it turns UP having moved RIGHT the previous move, and uniquely determines $< b_i >$ by the x coordinates at the points where it turns UP after having moved RIGHT. This gives a bijection between 321 avoiding permutations and lattice walks as required.

19. Permutations of n for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (also called 312 avoiding permutations).

123 132 213 231 321

Let the answer be f(n) for a particular n.
Note that here the notion of a permutation of n numbers is not important for f(n) to be the number of such permutations of n distinct numbers. If we took a random sequence of n distinct numbers and imposed the above condition, we'd get the same number of possibilities as if we considered permutations of [n]. This is important for the rest of the proof.
Now, let the least number be at index p. All numbers in the range 1 to p-1 must be lower than all numbers in the range p+1 to n. Hence, the p-1 lowest numbers (after this particular number) fill up the p-1 slots to the

left of p, and the remaining n-p numbers fill up the slots to the right. The numbers on the left and the numbers on the right have f(p-1) and f(n-p) possibilities to be arranged (based on the observation above that the sequence need not be a permutation for f to give the correct answer. Also note that the n-p elements on the right can be arranged independently of the p-1 elements on the left). Obviously, we get the Catalan recursion. All that is left is to prove the base case.

20. Sequences a_i of n integers such that i $\leq a_i \leq n$ and if $i \leq j \leq a_i$, $a_j \leq a_i$.

123 133 223 323 333

Proof: Consider a_i to be the number of elements lesser than the current element amongst the numbers in front of it in a 321-avoiding permutation. We can construct the required permutation as follows: consider the sequence $< a_i >$. Add the $a_i + 1$th smallest number which has not already been taken to the start of the list we are making. Since $a_i \leq n - i$, and there are always $n - i + 1$ elements remaining when we are at index i, this is always possible to do.
We need to show that the sequence a_i cannot result in a permutation which has a decreasing subsequence of length 3. FTSOC, let us assume there exists $i < j < k$ with $c_i > c_j > c_k$. Let i be minimal - that is, every element before the ith index is less than c_i and let j be minimal for this choice of i,k - that is, every element between index i and j is greater than c_i. Thus, $a_k = 0$, $a_i = c_i - i$ and $a_j = c_i - j$. Note that the condition $j - i > a_i - a_j$ is independent of the elements between indices i and j (can be seen by induction on the number of non-zero elements in the sequence a_i between i and j).
We need:

$$j - i > c_i - i - c_i + j$$

$$j - i > j - i$$

This is a contradiction. Hence, all permutations thus generated are 321-avoiding. We have defined an appropriate bijection.

21. Sequences $< a_i >$ of n integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r < j$.

111 112 113 121 123

Proof: Consider $b_i = a_i - 1$ to be the number of elements greater than the current element amongst the numbers before it in a 312-avoiding permutation. We can construct the required permutation as follows: consider the

sequence $< b_i >$ from back to forward. Add the $b_i + 1$th largest number which has not already been taken to the start of the list we are making. Since $b_i < i$, and there are always i elements remaining when we are at index i, this is always possible to do.
We need to show the resultant permutation is 312-avoiding. Suppose not. Suppose $c_i > c_k > c_j, i < j < k$. We assume WLOG that j is the minimal possible for a given i, k. That is, for $i < j' < j$, $c_{j'} > c_k$. For a particular j,k we also assume i is maximal. That is, for $i < i' < j$, $c_{i'} < c_i$. Thus, if $l = a_i$, $a_j = l + j - i$. Let the number of elements between j and k which are greater than c_k be x. $a_k = l + j - i + x$. We need $a_j - j \le a_k - k$. Therefore,

$$l + j - i - j \le l + j - i + x - k$$

$$k \le j + x$$

This is obviously not possible by the definition of j, k and x. Hence, this sequence cannot result in a permutation that is not 312-avoiding.
Every 312-avoiding permutation creates a distinct sequence above, and every sequence above generates a distinct 312-avoiding permutation. Hence we are done.

22. Permutations of the multiset $(1, 1, 2, 2, \cdots, n, n)$ so that the indices of the first occurences of $1, 2 \cdots n$ appear in increasing order and there is no subsequence of the form abab, where a and b are two distinct numbers from [n].

    112233 112332 122331 123321 122133

    Proof: Replace the first instance of each number with a (and the second instance with a). The second condition says there is no intersection of the arcs between the first and second instance of each number, hence this replacement generates a regular bracket sequence. Similarly, every regular bracket sequence can be translated into such a sequence of numbers. When meeting a (, replace it with the lowest number not already taken. When meeting a), replace it with the number corresponding to its unique left-bracket.

23. Stack-Sortable permutations - The permutations of n which can be sorted using an algorithm which uses a single stack:

    - Initialize an empty stack
    - For each input value x, while the stack is nonempty and x is larger than the top item on the stack, pop the stack to the output, and then push x onto the stack
    - While the stack is nonempty, pop it to the output

123 132 213 312 321

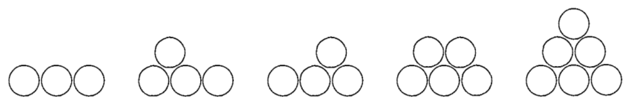
Proof: We claim that this algorithm fails if and only if the input is a 231-containing permutation. Let us assume it is a 231-containing permutation, then there exist indices i ¡ j ¡ k with $c_j > c_i > c_k$. Let j be such that it is minimal for a particular set of i,k. That is, every number from index i+1 to j is less than c_i. Then when c_j is processed, c_i is popped, but before c_k ever is processed. Hence, in the output, c_i and c_k are swapped.
Now, we need to show that a permutation without a subsequence of the form 231 gives the correct output. We accomplish this by induction on the index of the maximum element, along with on n itself. Obviously, when n=1, the hypothesis is trivial. We claim P(n,i)implies P(n,i+1) and P(n,i) holding for all i implies P(n+1,1).
The second claim is easy to show. If the maximum is in the first position, it is never popped until the end. The other n-1 elements can be viewed as a permutation which get sorted and popped, and finally the maximum is popped, resulting in a sorted array.
The second claim hinges on the fact that since the permutation is 231 avoiding, all the numbers in indices 1 to i are less than the numbers in the indices after i+2. When in the (i+1)th step, the maximum is added, it pushes out all the remaining elements in the exact same order as they would have been pushed out if the algorithm had terminated (i.e step 3 were reached) due to the permutation ending. Hence, the first i elements are returned in sorted order. Now, the later n-i elements get sorted and pushed out without affecting or being affected by the maximum. The maximum is pushed out last, hence a sorted array emerges.
The number of 231-avoiding permutations of n is the same as the number of 312-avoiding permutations (simply replace each element a_i with $n+1-a_i$). Hence, we are done.

24. Standard Young Tableaux of shape (n,n)

123	124	125	134	135
456	356	346	256	246

Proof: Let the numbers on the top column of the Tableau be the indices of (and the numbers on the bottom column be the indices of). This creates a bijection between standard young tableaux and regular bracket sequences of length 2n.

25. Ways to stack coins, with n consecutive coins in the bottom row.



Let us create a sequence of length n, with a_i being the number of coins if we start from the ith coin in the bottom row and proceed diagonally towards the left. For instance, the above picture represents 111,121,112,122 and 123. We observe the conditions $1 \leq a_i \leq i$ and $a_i + 1 \geq a_{i+1}$. We have already seen this before, and observed that the number of non-decreasing sequences $i - a_i + 1$ can be put in bijection with lattice paths that do not cross the main diagonal from below.