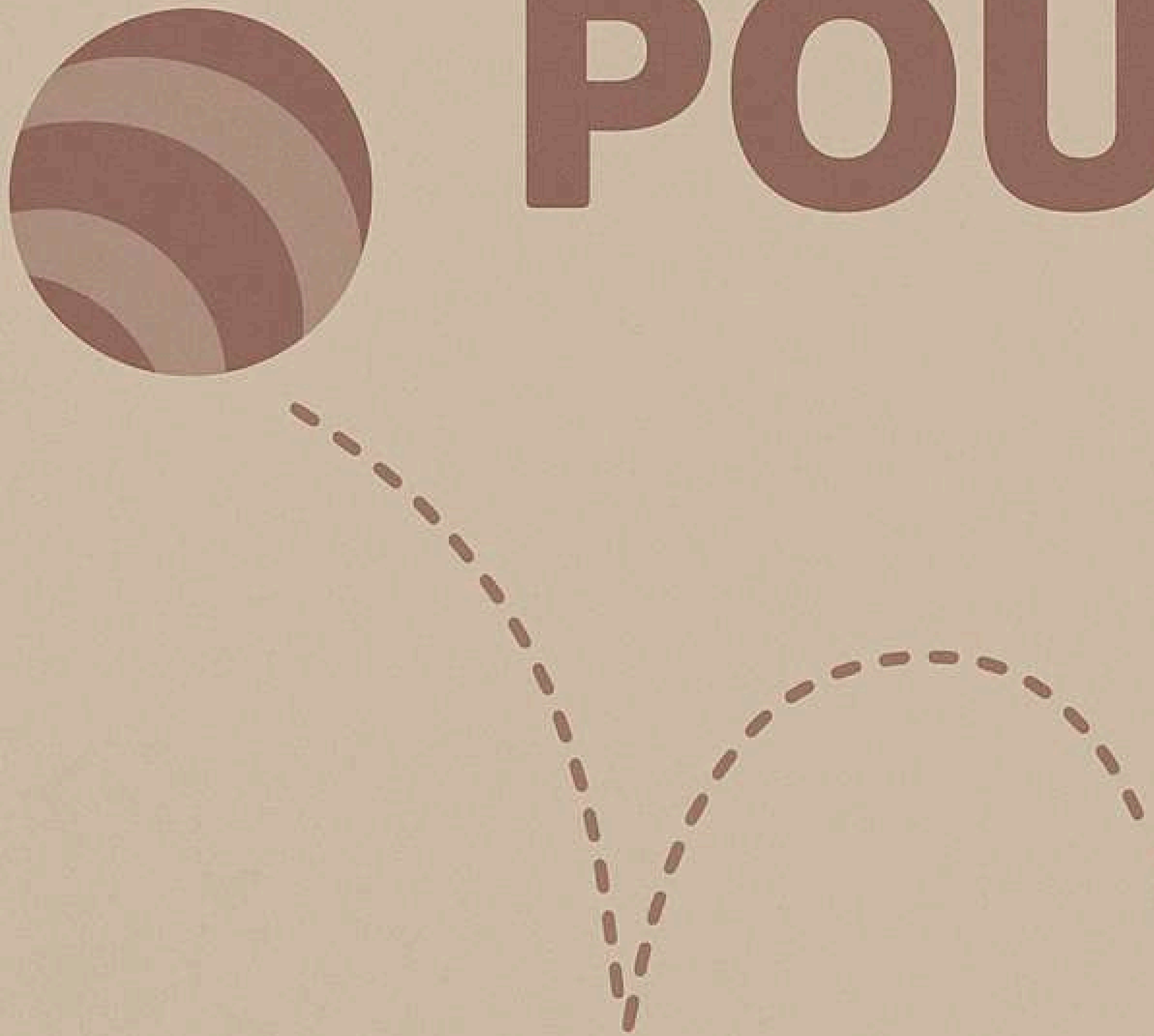
MATHS N PHYSICS
IIT BOMBAY

BOUNCE AND POUNCE

itc Institute Technical Council



RULES

Pounce: All teams except the one to which the Question is directed can "pounce" on it (submit their answer on paper) within 120 seconds of its release. Scoring is +30 on a correct answer and -15 on a wrong answer.

Bounce: After pounce is closed, bounce starts. Team 'i' answers directly within an additional time of 30 seconds; they get +20 if they're correct and 0 otherwise. If they get it wrong, it goes to the next team. Teams that pounce don't get to answer on a bounce. This goes on until the question is answered or it comes back to team 'i' again. The round proceeds in cyclic order thereafter.

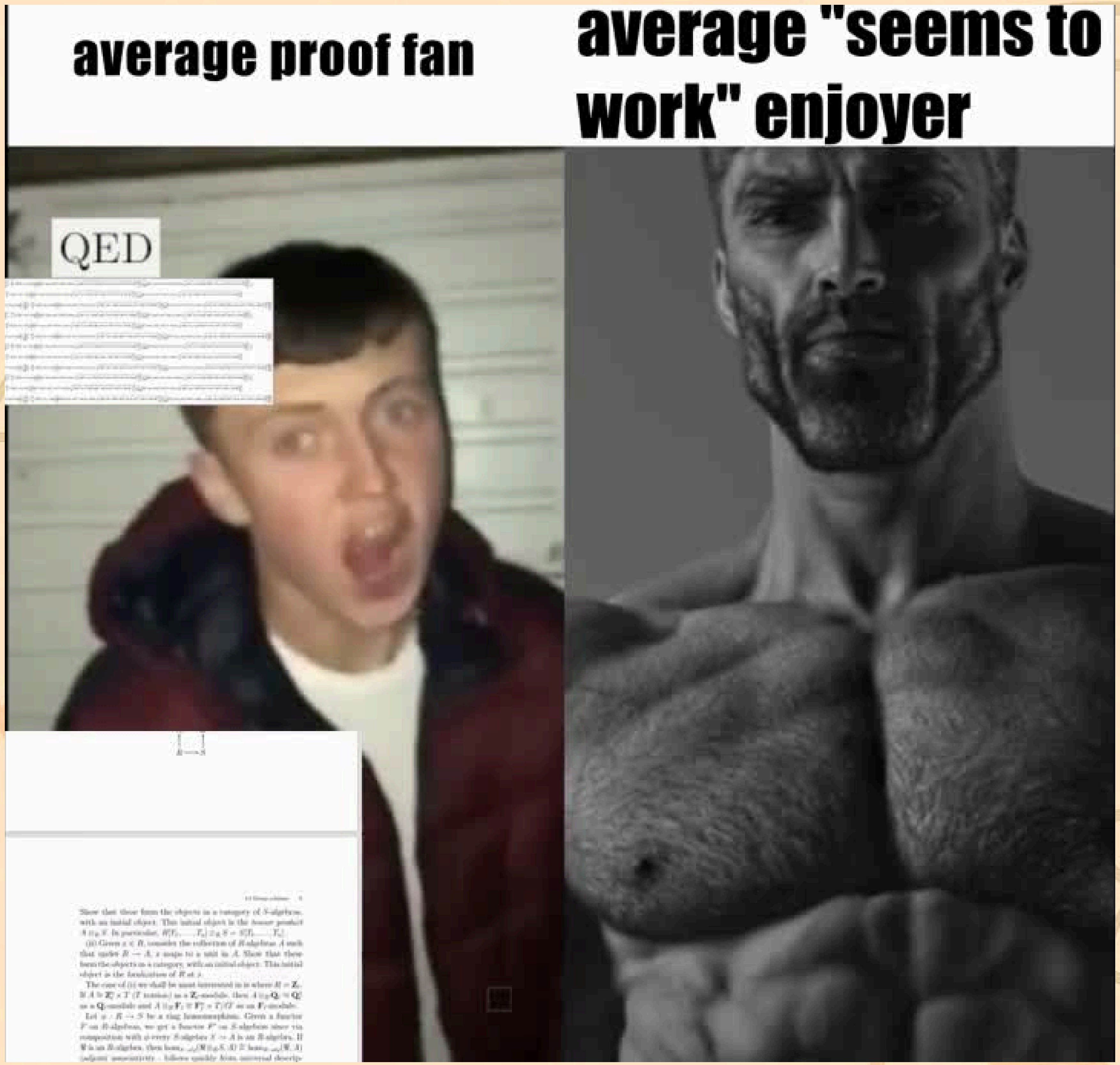
ROUND 1

QUESTION 1

Given a permutation of length n has more than 1 fixed point, let A_n be the expected number of fixed points, Find

$$\lim_{n\to\infty} A_n$$

SAFETY SLIDE



ANSWER

$$\frac{e-1}{e-2}$$

safety slide

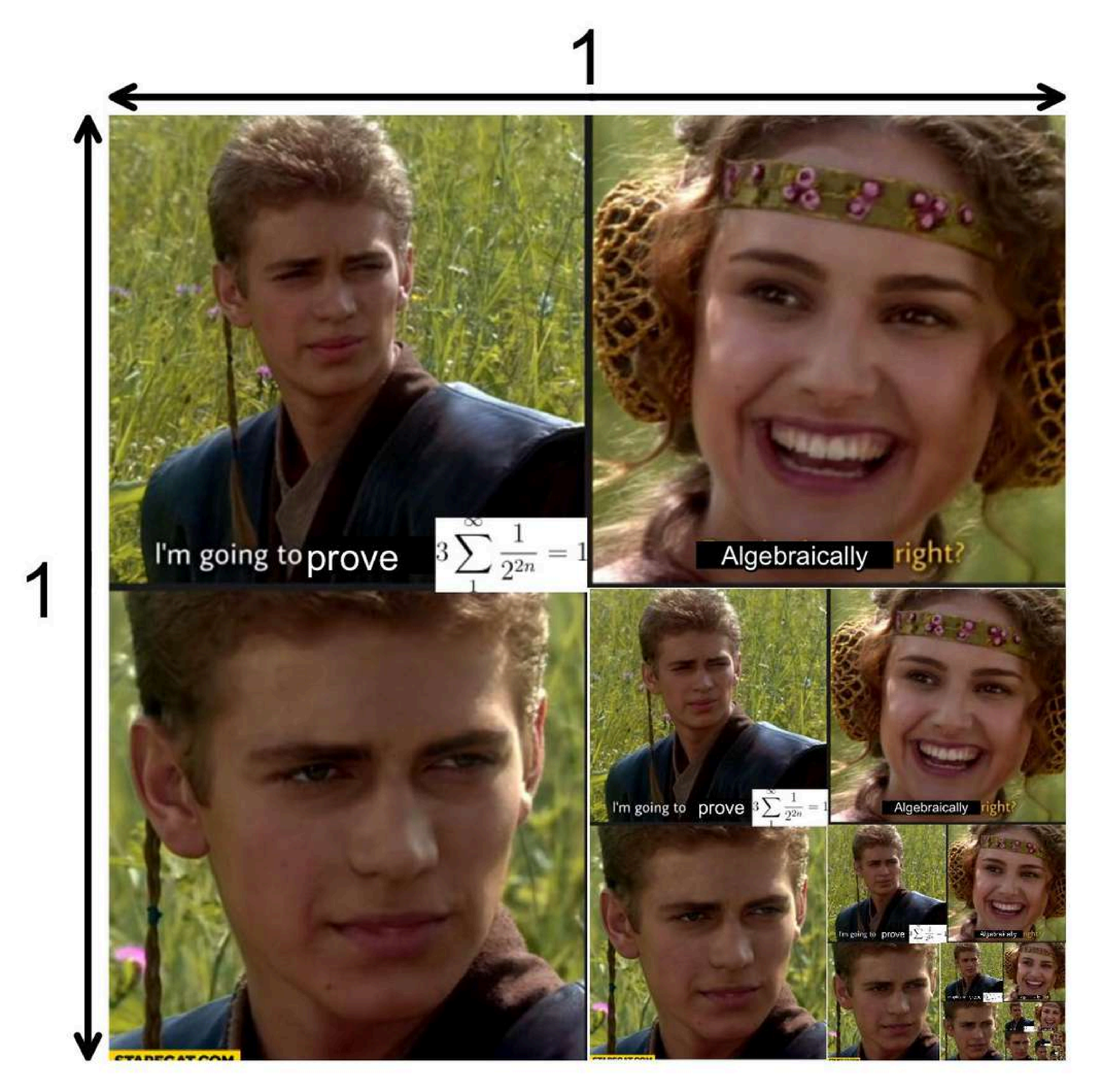
QUESTION 2

There is a new scoring system for numbers via iterated exponentiation denoted by Θ_n where for an n digit number $A_1 A_2 \ldots A_n$ ($A_1, \ldots, A_n$ are digits of the number in base 10), the score is calculated as

$$\Theta_n(A_1 \ldots A_n) = A_1^{A_2^{\cdot^{\cdot^{\cdot^{A_n}}}}}$$

Let S be the set of 9 digit numbers $A_1 \ldots A_9$ with distinct digits that maximize the value of $\Theta_9(A_1 \ldots A_9)$. Find the sum of elements of S.

SAFETY SLIDE



ANSWER

234567891

safety slide

QUESTION 3

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies for all pairs of reals x, y

$$xf(x + f(y)) = (y - x)f(f(x))$$

If $f(5) = 0$, find the value of $f(11)$

SAFETY SLIDE

ANSWER

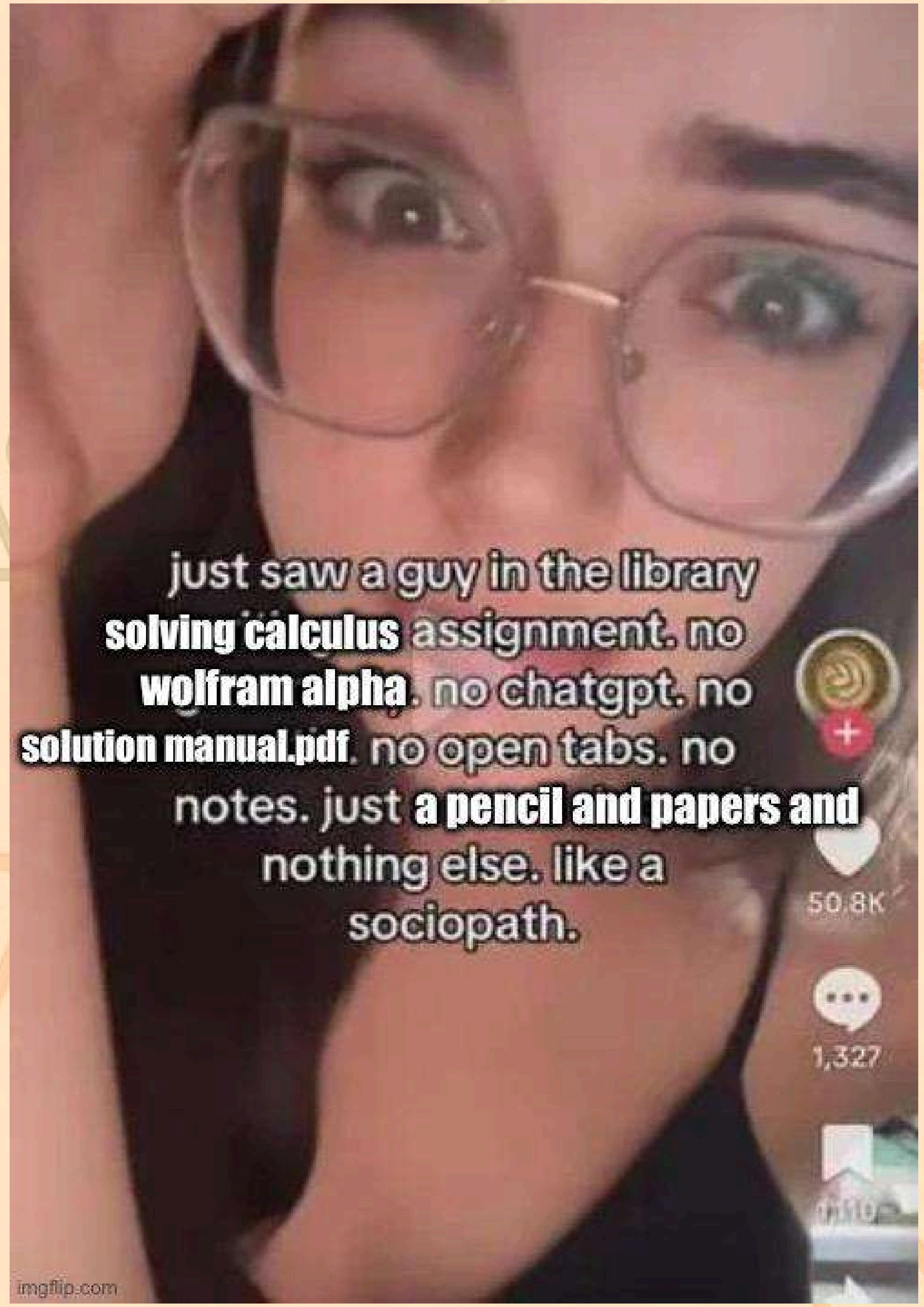
$$-6$$

safety slide

QUESTION 4

An autobiographical number is one where the first digit describes how many 0's it has, the second digit describes how many 1's it has, and so on, so that the (n+1)’th digit describes how many n’s it has. For example 1210 is an autobiographical number because it has 1 zero, 2 ones, 1 two and 0 threes. Find a ten digit autobiographical number.

SAFETY SLIDE



ANSWER

6210001000

safety slide

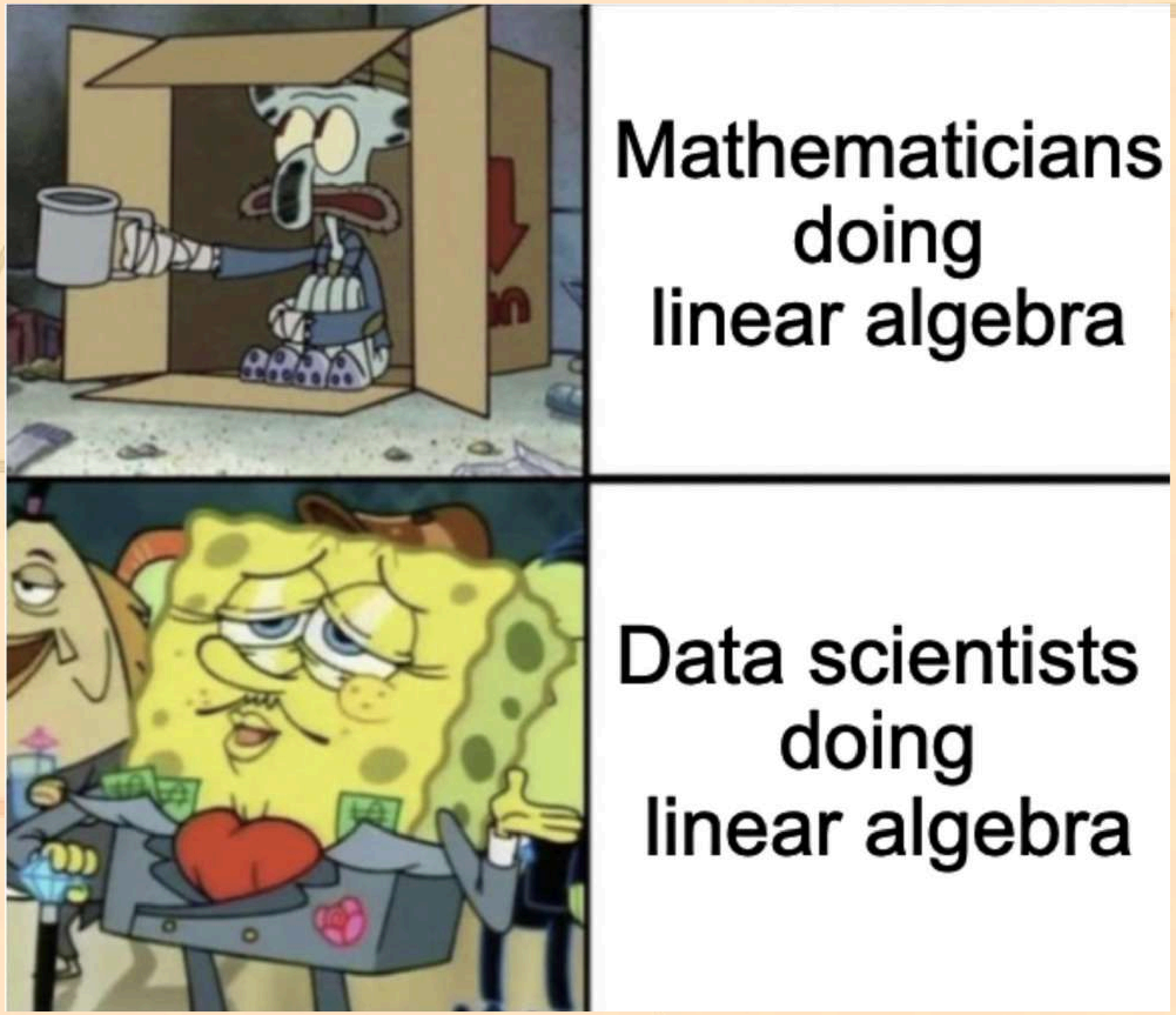
QUESTION 5

f is a continous real valued function defined on all reals such that

$$f(2x+3) = f(x+1) + f(x+2) - 2x^2 - 6x - 4$$

Given that $f(0) = 0, f'(0) = 0$, Find the value of $f(3)$

SAFETY SLIDE



ANSWER

69

safety slide

QUESTION 6

Let k be the largest integer that cannot be expressed as $am + bn$, $a, b \geq 0$ where

$$m = 2^{2024} + 1, n = 2^{2025} + 1$$

Find $\lceil \log_2(k+1) \rceil$

SAFETY SLIDE

ANSWER

4049

safety slide

ROUND 2

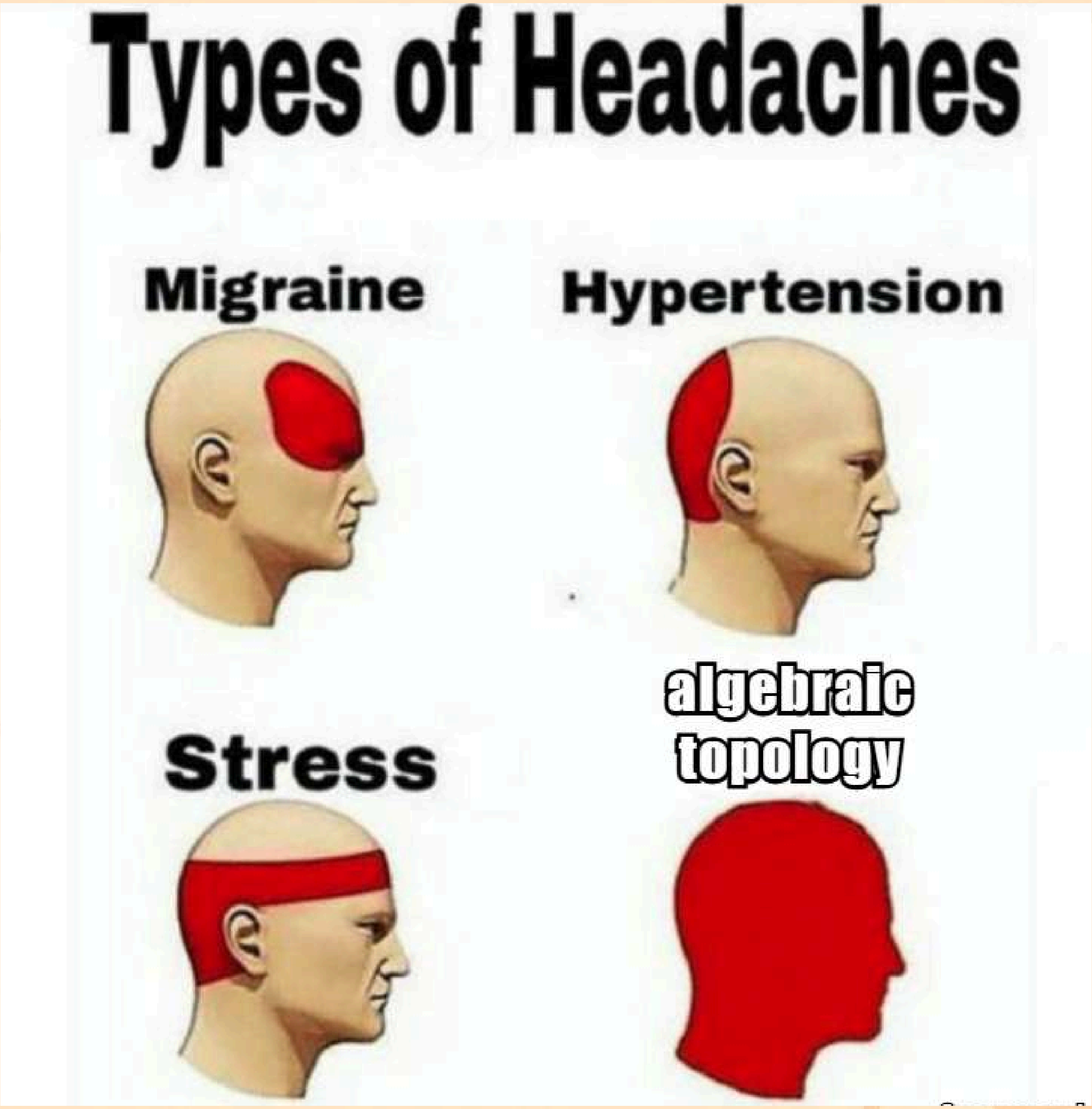
QUESTION 7

Let a, b, c be integers that satisfy $ab + bc + ca = 1$ and the value of $|a| + |b| + |c|$ is the closest possible value to 2025. Find the difference between

$$(1 + a^2)(1 + b^2)(1 + c^2)$$

and the nearest perfect square that is less than or equal to it.

SAFETY SLIDE



ANSWER

$$0$$

$(1+a^2)(1+b^2)(1+c^2)$ **must be a perfect square**

safety slide

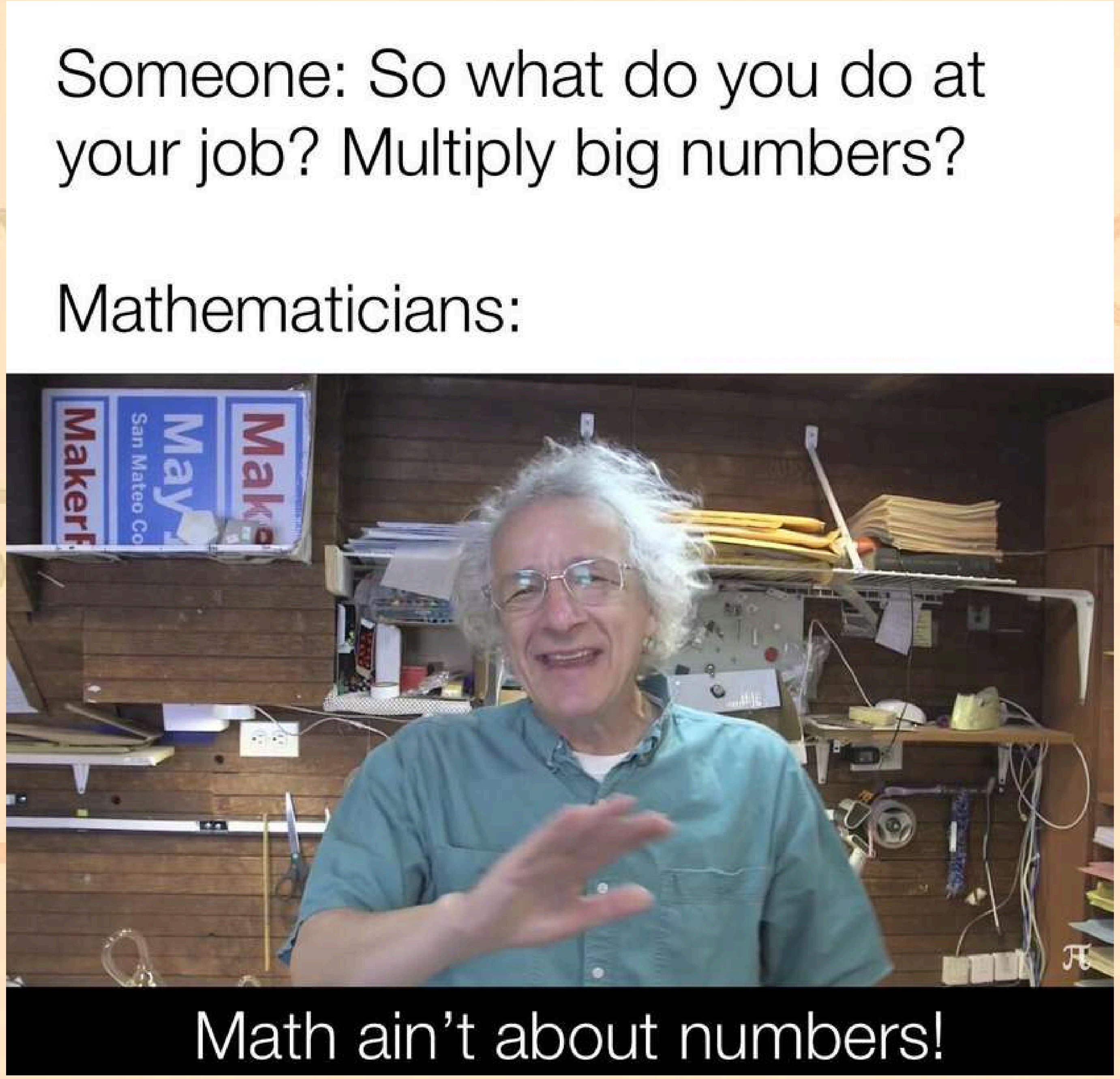
QUESTION 8

Find the limit of

$$S_n = \log_e\left(\sqrt[n^2]{1^1 \cdot 2^2 \cdots n^n}\right) - \log_e\left(\sqrt{n}\right)$$

as $n \to \infty$

SAFETY SLIDE



ANSWER

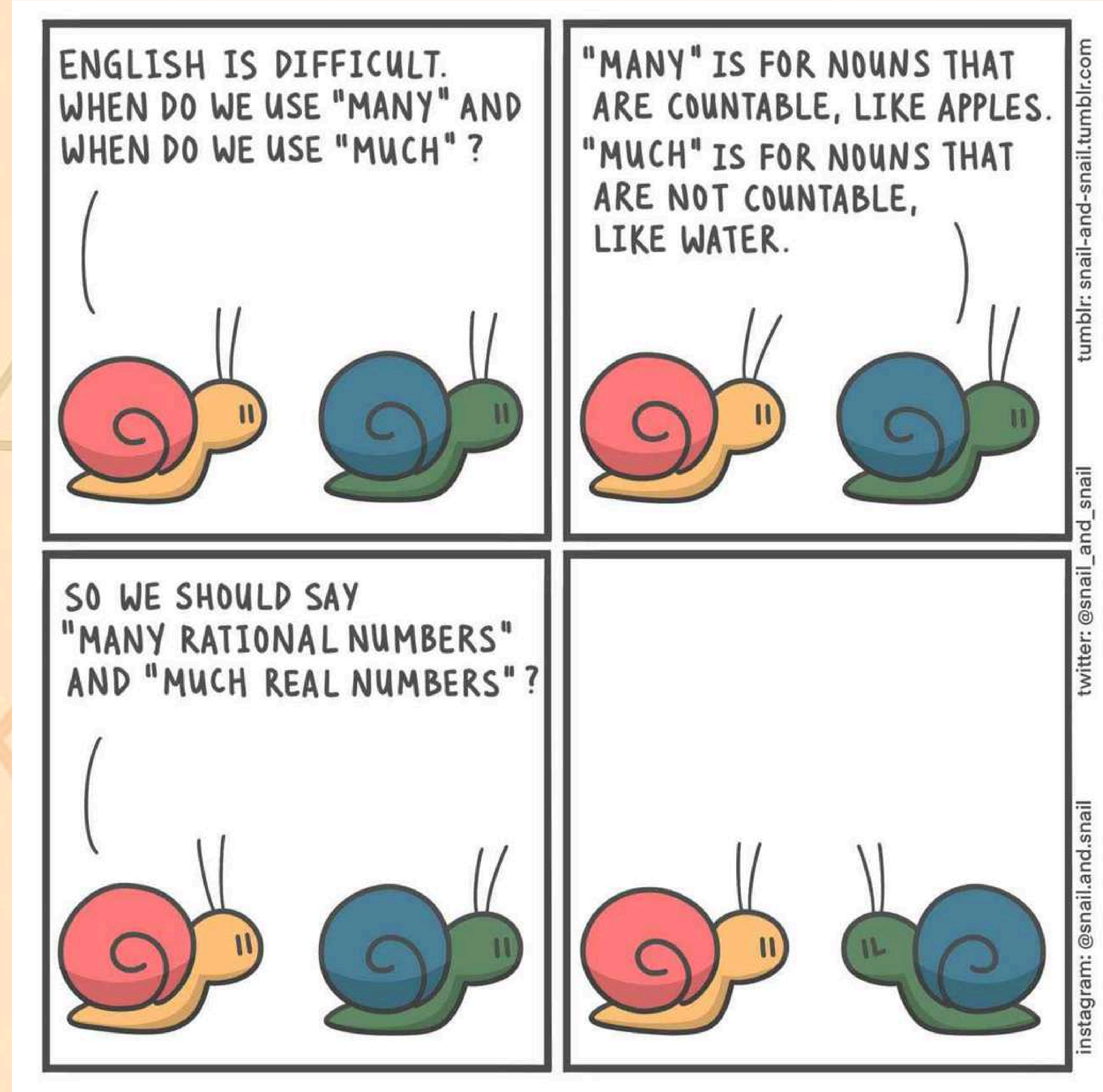
$$-\frac{1}{4}$$

safety slide

QUESTION 9

Determine all integers n for which the number 11111 in base n is a perfect square.

SAFETY SLIDE



ANSWER

3

safety slide

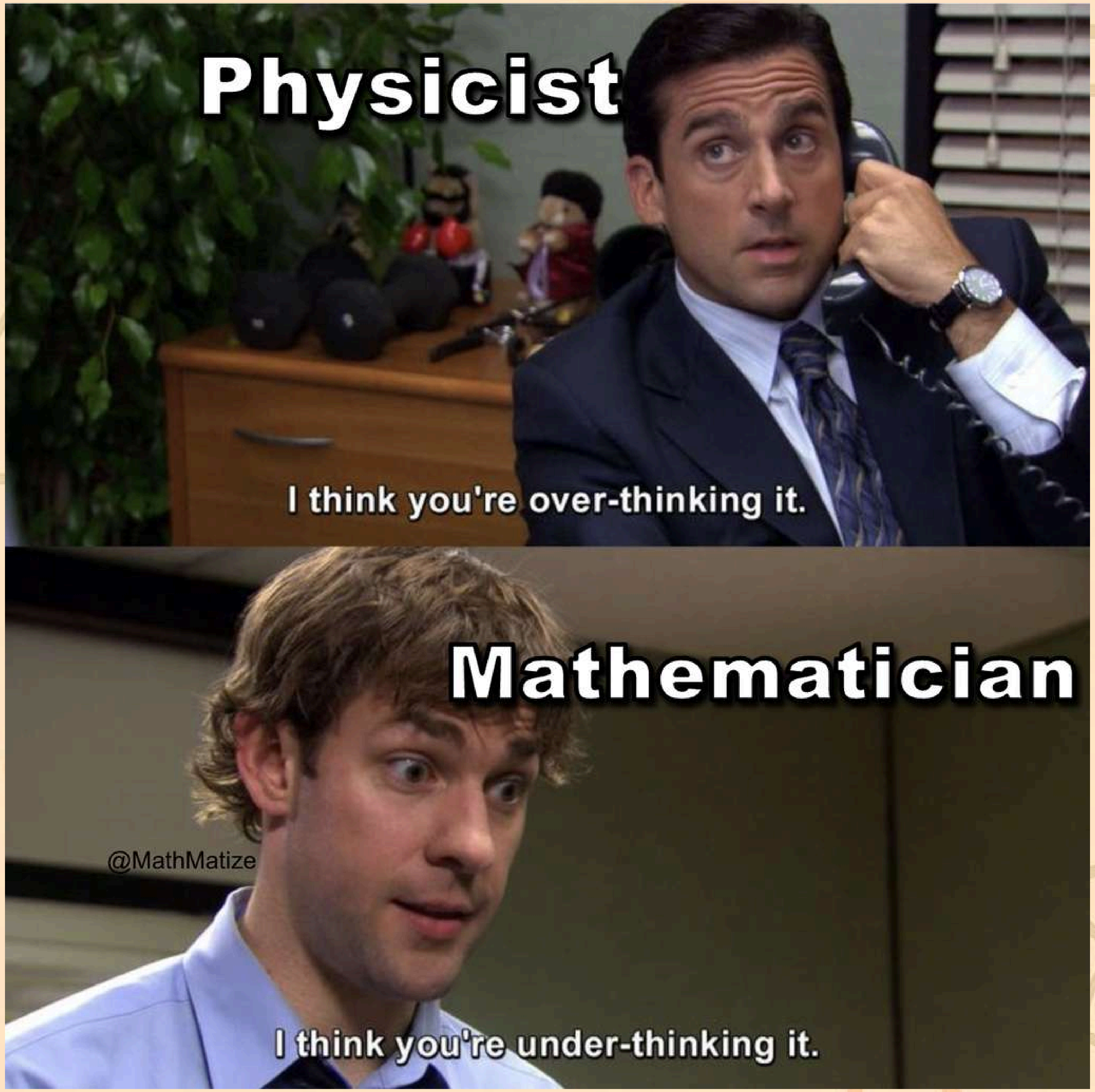
QUESTION 10

Alice and Bob play a game on a 5x5 grid, where they take turns to choose some box, and remove all boxes in its row or column that are not already removed, with Alice going first. Initially they are told that they will get 1 point per box removed, and they played optimally according to that. Bob wins in the case of a tie.

However, then Christopher notices that this game is very biased towards Alice, so decides to change the scoring system, such that Bob wins. Now Alice and Bob get a, b, c, d, e points per box they had removed in the 1st, 2nd, 3rd, 4th and 5th round respectively. If all $a, b, . . . , e$ are positive integers in arithmetic progression, what is the maximum value of

$$e/a$$

SAFETY SLIDE



ANSWER

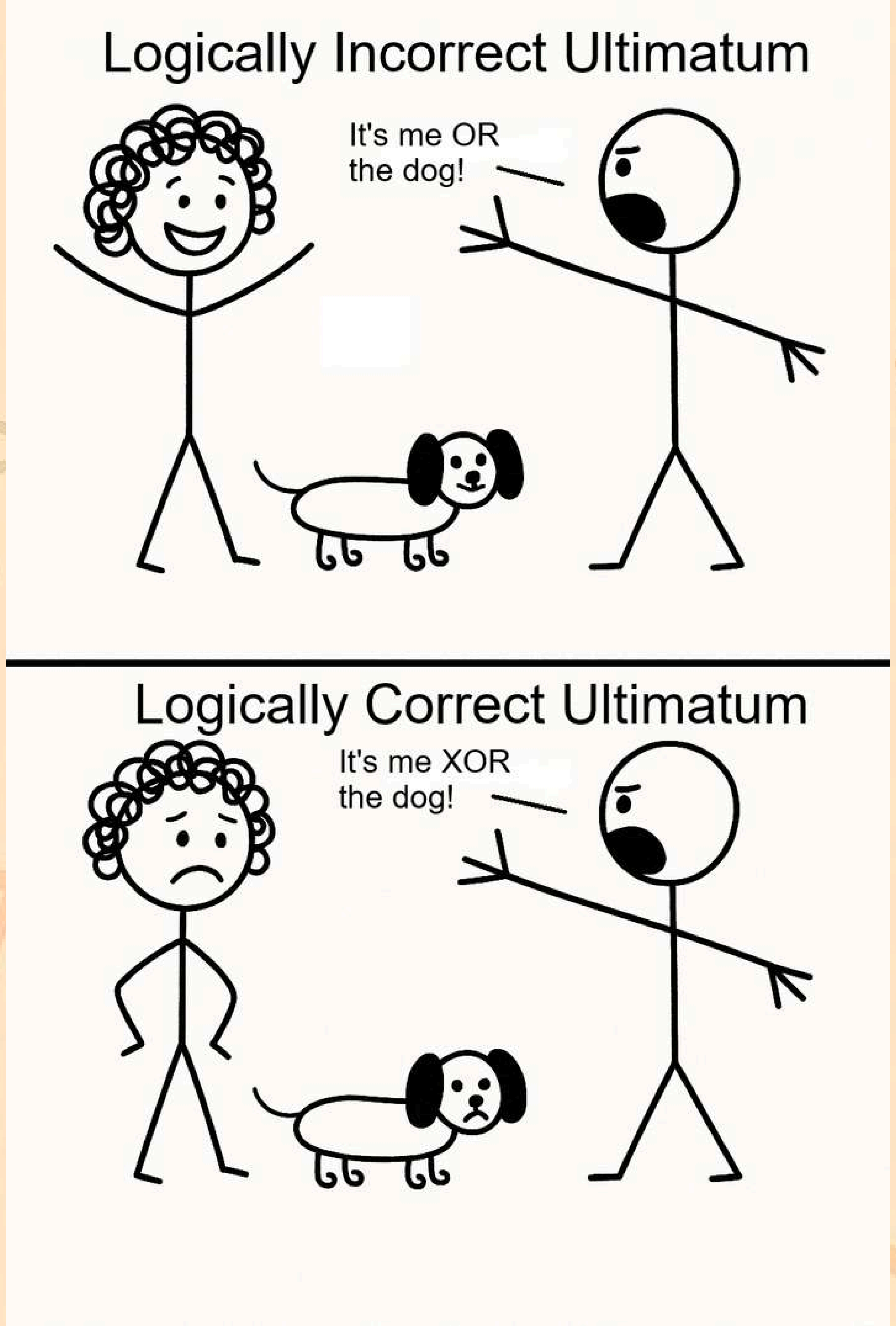
11

safety slide

QUESTION 11

Consider the lens-shaped region bounded by the two parabolas in the XY plane $y = x^2$ and $y = 4 - x^2$ What is the maximum number of sides for a regular polygon whose vertices all lie on the perimeter of this lens?

SAFETY SLIDE



ANSWER

8

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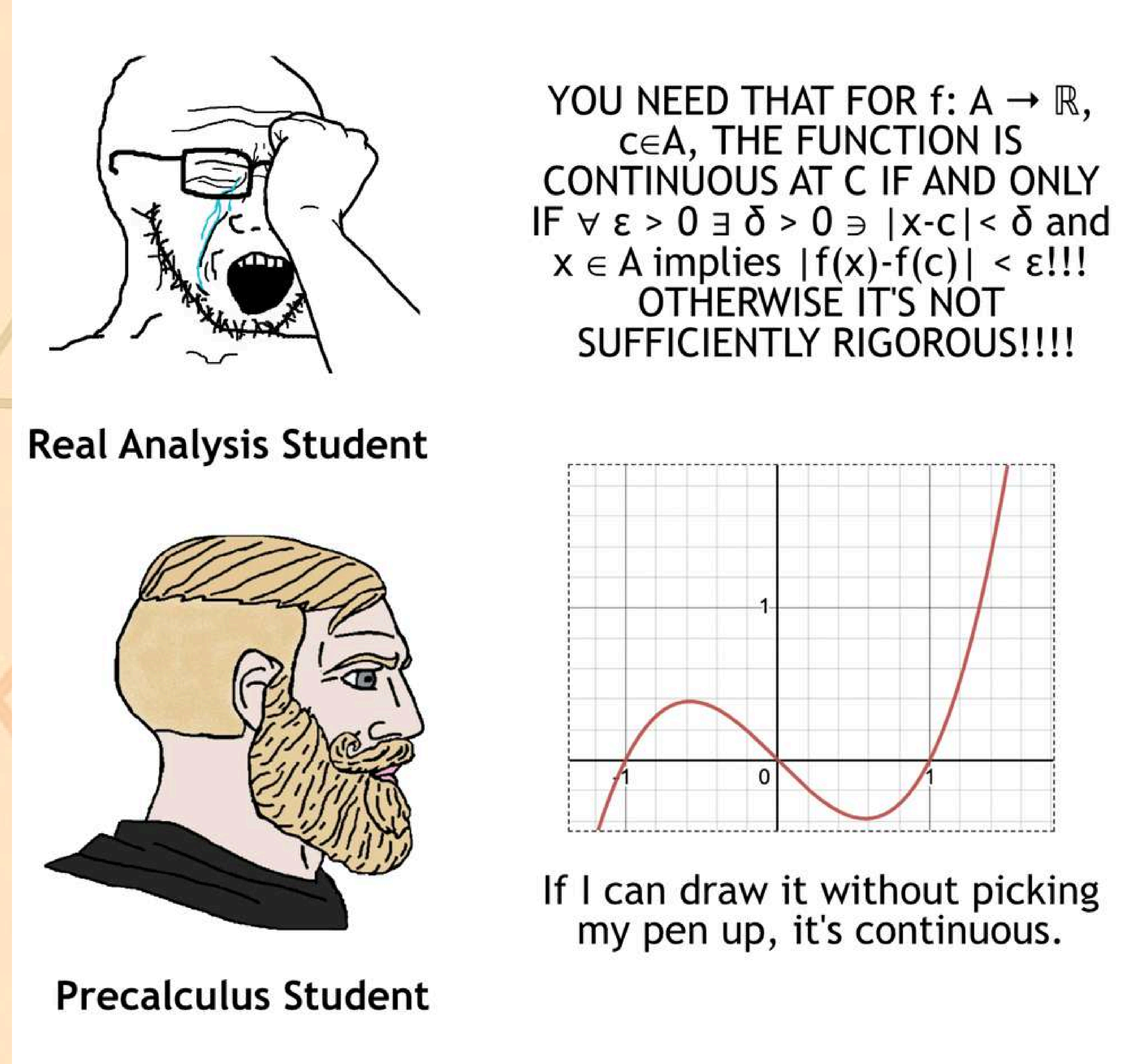
QUESTION 12

Let f be a continuous function $f : [0,1] \to \mathbb{R}$ satisfying $f(x) + f(y) \geq |x - y| \; \forall x, y \in [0,1]$

Find the minimum of $\int_0^1 f(x)dx$ over all such functions

SAFETY SLIDE



ANSWER

$$\frac{1}{4}$$

safety slide

AUDIENCE QUESTIONS

QUESTION

If you take any 4 digit number, rearrange its digits to form the maximum and the minimum possible numbers and take their difference, and keep repeating this process, you will reach a constant within a finite number of iterations. what is the value of this constant? What is it called, and what is the upper bound on the number of iterations it takes to reach this constant

ANSWER

6174, Kaprekar constant

it takes at most 7 iterations to reach

QUESTION

A monic polynomial $P(x)$ is such that for all natural numbers y there is a natural number x such that

$$P(x) = 2^y$$

Find all possible degrees of the polynomial

ANSWER

1

QUESTION

Let $\{a_n\}_{n\geq 0}$ be a sequence of reals such that

$$\sum_{i=1}^{\infty} \frac{a_i}{i} = e$$

Find the value of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i$$

ANSWER

0

THE END